Writing Down Tables of Characters for Cyclic Point Groups

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In textbooks on point groups the usefulness of cyclic point groups is often neglected. Nevertheless, these groups have a very interesting property: the characters of their representations may be written readily, almost automatically. Applications to S_n groups are given as examples.

C_n Groups

It is well known by students that C_n groups are abelian and cyclic. Let us now choose an arrangement in writing down the elements which is slightly different from the usual one. Rather than starting with the identity, we will begin with the element generating the groups and its powers; all elements of a given C_n group are generated by a given element and its powers. Since the group is abelian there are as many irreducible representations as elements (the order of the group: n). The table of characters arranged in such a way will have characters for the identity element in the last column. An element in the p column is the p power of the element in the first column. Corresponding characters have the same relationship.

$$\chi(C_n^p) = [\chi(C_n)]^p$$

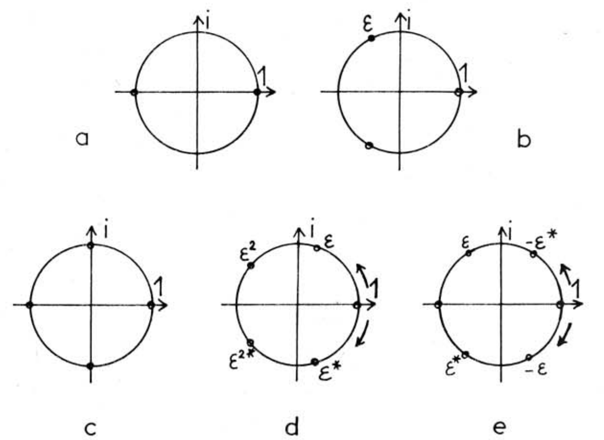
For the last column (identity element)

$$1 = \chi(C_n^n) = [\chi(C_n)]^n$$

It appears that possible characters for the generating elements are the nth roots of unity (1). There are n such values (including 1) in group C_n. Thus, the tables of characters can be written out straightforwardly by using the "complex circle" representation.

C_2	C_2	$C_2^2 = E$	(Fig. 1a)
A	1	1	
B	−1	1	

C_3		C_3	C_3^2	$C_3^3 = E$	(ϵ = exp $2\pi i/3$)	(Fig. 1b)
A		1	1	1		
E	{	ϵ	ϵ^*	1		
		ϵ^*	ϵ	1		



Small circles indicate possible characters in complex representation when there are 2, 3, 4, 5, or 6 elements in the cyclic group.

C_4	C_4	C_4^2	$C_4^3 = E$	$C_4^4 = E$	(Fig. 1c)
A	1	1	1	1	
(E)	i	−1	$-i$	1	
B	−1	1	−1	1	
(E)	$-i$	−1	i	1	

In this last example, *the usual arrangement of representations is lost.* for the sake of representation symbols, it is useful to keep the conjugated roots in pairs. This merely means that the "complex circle" should be followed by 2 symmetrical ways: clockwise and counterclockwise. For example, if one wishes to write the table of characters for C_5 in a way that is not too different from the usual one, the roots may be classified as follows:

$$1,\ e^{i2\pi/5} = \epsilon,\quad e^{(-i2\pi/5)} = \epsilon^*,\quad e^{2i2\pi/5} = \epsilon^2,\quad e^{-2i2\pi/5} = \epsilon^{2*}$$

This gives the following table for C_5

[1] See Mc Weeny. "Symmetry," Pergamon, **1963,** p. 92.

		C_5	C_5^2	C_5^3	C_5^4	$C_5^5 = E$
A		1	1	1	1	1
E_1	{	ϵ	ϵ^2	ϵ^{2*}	ϵ^*	1
		ϵ^*	ϵ^{2*}	ϵ^2	ϵ	1
E_2	{	ϵ^2	ϵ^*	ϵ	ϵ^{2*}	1
		ϵ^{2*}	ϵ	ϵ^*	ϵ^2	1

S_n Groups

The S_n groups are cyclic abelian groups, too. A detailed study is very useful for tutorial purposes.

Group S_1 appears to be identical to C_s and isomorphic to C_2 group (Table 1a and Fig. 1a).
Group S_2 ($\equiv C_i$) is isomorphic to S_1 and C_2(Table 1b and Fig. 1a).
Group S_3 ($\equiv C_{3h}$ group) has 6 elements and is isomorphic to C_6 (Table 1c and Fig. 1e).
Group S_4 is isomorphic to C_4 (Table 1d and Fig. 1c).
Group S_5 ($\equiv C_{5h}$) is isomorphic to C_{10} (Table 1e).
Group S_6 has 6 elements and is isomorphic to C_6 (and S_3!) (Table 1f and Fig. 1e).

(a)

$S_1 = C_s$	$S_1 = \sigma$	$S_1^2 = E$
A'	1	1
A''	−1	1

(b)

$S_2 = C_i$	$S_2 = i$	$S_2^2 = E$
A_g	1	1
A_u	−1	1

(c)

$S_3 = C_{3h}$		S_3	$S_3^2 = C_3^2$	$S_3^3 = \sigma_h$	$S_3^4 = C_3$	S_3^5	$S_3^6 = E$
A'		1	1	1	1	1	1
E''	{	$-\epsilon^*$	ϵ	−1	ϵ^*	$-\epsilon$	1
		$-\epsilon$	ϵ^*	−1	ϵ	$-\epsilon^*$	1
				etc.			

(d)

S_4		S_4	$S_4^2 = C_2$	S_4^3	$S_4^4 = E$
A		1	1	1	1
E	{	i	−1	$-i$	1
		$-i$	−1	i	1
B		−1	1	−1	1

(e)

$S_5 = C_{5h}$	S_5	$S_5^2 = C_5^2$	S_5^3	$S_5^4 = C_5^4$	$S_5^5 = \sigma_h$
	$S_5^6 = C_5$	S_5^7	$S_5^8 = C_5^3$	S_5^9	$S_5^{10} = E$
			etc.		

(f)

S_6	S_6	C_3	$S_6^3 = i$	C_3^2	S_6^5	$S_6^6 = E$
	same as $S_3 = C_{3h}$					

C_{nh} groups with odd values of n appear to be cyclic. Writing S_n tables of characters in such a way is almost automatic. For students it has other advantages. They understand more clearly what exactly an S_n operation is, there is no longer any need to calculate multiplication table for S_n groups, and isomorphism is well demonstrated. There is a drawback: direct product features of some groups disappear from the resulting tables; but this aspect can be recovered easily as explained below.

More Tutorial Applications

(1) Multiplying together different cyclic groups such as C_n, S_n, C_i and C_s with their axes and planes parallel or perpendicular to each other, it is possible to generate readily character tables for many more point groups.

(2) A quick examination of the S_n^p operations shows that if p is even $S_n^p = C_n^p$ and if p is odd S_n^p equals the product of C_n^p by σ_h. So if one picks from list $\{S_n^p\}$ all the elements having p even (and this is very quickly done by picking one in every second operation), one gets a set of proper operations which is a group, the order of which is n if n is odd and $n/2$ is n is even. The remaining elements obviously form a coset.

If σ_h exists in the coset, the whole S_n group appears as the product of group C_n by group C_s. This enables another more traditional way of writing the table. It happens if n is odd, i.e., for all C_{nh} groups having n odd. If $n = 4k + 2$ (k integer) (for example n = 2, 6, 10) the same procedure picking one in every second operation yields the C_{2k+1} proper group. Inversion remains in the coset which may be obtained by multiplying C_{2k+1} by i. The table is obtained from it. But S_n groups having $n = 4k$ cannot be decomposed in a similar way. Usually they are dealt with in a lengthy and awkward manner. Considering them as cyclic is much easier and unifying.

(3) The method is a good introduction to 1-, 2-, or 3-dimensional translational groups and Born cyclic conditions.

Summary

Cyclic properties give a unified view of C_n, S_n, and one-dimensional translational groups and an efficient way to write down corresponding tables of characters.