SCRITTO DEL 18/9/17

1. (a) Sia $M = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} \in GL_2(\mathbb{R})$.

Sia $\varphi : GL_2(\mathbb{R}) \longrightarrow \mathbb{R}^4$

$$\varphi\left(\overline{X} = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}\right) = (x_{11}, x_{12}, x_{21}, x_{22})$$

il sistema di coordinate standard in $GL_2(\mathbb{R})$.

$$\Rightarrow (\varphi \circ L_M \circ \varphi^{-1})\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \varphi\left(\begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) =$$

$$= \varphi\begin{pmatrix} m_{11} \cdot a + m_{12} \cdot c & m_{11} \cdot b + m_{12} \cdot d \\ m_{21} \cdot a + m_{22} \cdot c & m_{21} \cdot b + m_{22} \cdot d \end{pmatrix} =$$

$$= (m_{11} a + m_{12} c,\ m_{11} b + m_{12} d,\ m_{21} a + m_{22} c,$$
$$m_{21} \cdot b + m_{22} \cdot d)$$

$\Rightarrow L_M \in C^\infty$.

L_M è invertibile ed $(L_M)^{-1} = L_{(M^{-1})}$

$\Rightarrow L_M$ è un diffeomorfismo.

(b)

$$D_{I_2} L_M(A) = \frac{d}{dt}\Big|_{t=0} L_M(I_2 + t\cdot A) = \frac{d}{dt}\Big|_{t=0} t\cdot M\cdot A = M\cdot A$$

$$v_A : GL_2(\mathbb{R}) \to GL_2(\mathbb{R}) \times M_2(\mathbb{R})$$

$$v_A(M) = (M, M\cdot A)$$

è C^∞ $\Rightarrow$ è un campo vettoriale su $GL_2(\mathbb{R})$.

(d) $$[v_A, v_B] = \sum_{i,j=1}^{2} [v_A, v_B]_{ij} \cdot \frac{\partial}{\partial x_{ij}}$$

$$[v_A, v_B]_{ij} = [v_A, v_B](x_{ij}) = v_A(v_B(x_{ij})) - v_B(v_A(x_{ij}))$$

$$v_B(x_{ij}) = \,?$$

Notiamo: $v_B(M) = (M, \xi_B(M))$

$$\xi_B(M) = \sum_{k,l=1}^{2} (\xi_B(M))_{k,l} \frac{\partial}{\partial x_{k,l}} = \sum_{k,l=1}^{2} \underbrace{(M\cdot B)_{k,l}}_{\text{elemento di posto } k,l \text{ di } M\cdot B} \frac{\partial}{\partial x_{k,l}}$$

$$\Rightarrow v_B(x_{ij})(M) = (M\cdot B)_{i,j} = m_{i1}\cdot b_{1j} + m_{i2}\cdot b_{2j}$$

dove $B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$

$$\overbrace{v_A\Big(\underbrace{v_B(x_{ij})}_{\in C^\infty(GL_2)}\Big)}^{\in C^\infty(GL_2)}\big(\overset{GL_2}{\overset{\cup}{N}}\big) = v_A\big(m_{i1}b_{1j}+m_{i2}b_{2j}\big)(N) =$$

$$= \frac{d}{dt}\Big|_{t=0}\big(m_{i1}\cdot b_{1j}+m_{i2}b_{2j}\big)(N+t\cdot A) =$$

$$= \frac{d}{dt}\Big|_{t=0}\Big((n_{i1}+ta_{i1})\cdot b_{1j}+(n_{i2}+ta_{i2})\cdot b_{2j}\Big) = (A\cdot B)_{ij}$$

$$\Rightarrow [v_A, v_B](x_{ij}) = (A\cdot B)_{ij} - (B\cdot A)_{ij} = [A,B]_{i,j}$$

$$\Rightarrow \boxed{[v_A, v_B] = v_{[A,B]}}$$

(c) Basta sostituire $A = \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix}$

nella formula del punto (d).

— o —

2. (a) Usiamo <u>Mayer-Vietoris</u>, con

$$U := \mathbb{R}^3 \setminus \{P, Q\} \qquad V := D_1 \sqcup D_2,$$

dove $D_1 = D^3\left(P=(0,0,0), \frac{1}{4}\right)$ = disco di $\mathbb{R}^3$ (palla) di centro P e raggio $\frac{1}{4}$

$D_2 = D^3\left(Q=(1,0,0), \frac{1}{4}\right)$ = disco in $\mathbb{R}^3$ di centro Q e raggio $\frac{1}{4}$

$$\Rightarrow U \cup V = \mathbb{R}^3, \qquad U \cap V \underset{h}{\simeq} \underbrace{S^2\left(P, \tfrac{1}{4}\right) \cup S^2\left(Q, \tfrac{1}{4}\right)}$$

sfere 2-dimensionali di centro P, Q e raggio $1/4$

Mayer-Vietoris:

$$0 \to H^0_{dR}(\mathbb{R}^3) \to \underbrace{H^0_{dR}(U)}_{\overset{\shortparallel}{\mathbb{R}}} \oplus H^0_{dR}(V) \to H^0_{dR}(U \cap V) \to \underbrace{H^1_{dR}(\mathbb{R}^3)}_{=0} \to$$

$$\overset{0}{\to} H^1_{dR}(U) \oplus H^1_{dR}(V) \overset{0}{\to} \overbrace{H^1_{dR}(U \cap V)}^{=0} \to \overbrace{H^2_{dR}(\mathbb{R}^3)}^{=0} \to H^2_{dR}(U) \oplus H^2_{dR}(V)$$

$$\Rightarrow H^1_{dR}(U) = 0$$

$$\to \overbrace{H^2_{dR}(U \cap V)}^{\mathbb{R}^2} \to \underbrace{H^3_{dR}(\mathbb{R}^3)}_{=0} \to H^3_{dR}(U) \oplus H^3_{dR}(V) \to$$

$$\to \underbrace{H^3_{dR}(U \cap V)}_{0} \to 0$$

$$\Rightarrow H^k_{dR}(\underbrace{\mathbb{R}^3 \setminus \{P,Q\}}_{=U}) \cong \begin{cases} \mathbb{R} & k=0 \\ \mathbb{R}^2 & k=2 \\ 0 & k \notin \{0,2\} \end{cases}$$

(b) $$\bigoplus_k H^*_{dR}(\mathbb{R}^3 \setminus \{P,Q\}) = \underset{\underset{\mathbb{R}\langle 1 \rangle}{\shortparallel}}{H^0_{dR}(\mathbb{R}^3 \setminus \{P,Q\})} \oplus \underbrace{H^2_{dR}(\mathbb{R}^3 \setminus \{P,Q\})}_{\underset{\mathbb{R}\langle\alpha\rangle \oplus \mathbb{R}\langle\beta\rangle}{\shortparallel}}$$

1 è elemento neutro

$$\alpha \wedge \beta = 0 = \alpha \wedge \alpha = \beta \wedge \beta$$

(c) Non esiste una tale M.

Infatti, se esistesse, $H^k_{dR}(M) \cong \begin{cases} \mathbb{R} & k=0 \\ \mathbb{R}^2 & k=2 \\ 0 & k \notin \{0,2\} \end{cases}$

$\Rightarrow$ M sarebbe connessa, ma non compatta (altrimenti, per Poincaré, $H^2_{dR}(M) \cong H^0_{dR}(M)$).

~~Siccome~~ Ma M non compatto $\Rightarrow$ $\underbrace{H^0_{dR,c}(M)}_{\text{coomologia a supporto compatto}} = 0$

$$\Rightarrow H^2_{dR}(M) \cong 0 \quad \lightning.$$

3. (a) $d\omega = 4\, dx\wedge dy\wedge dz\wedge dw = 4\cdot \underbrace{\mathrm{Vol}_{\mathbb{R}^4}}_{\text{forma di volume standard di } \mathbb{R}^4}$

(b) $\displaystyle\int_{S^3} i^*\omega \overset{\cancel{\text{Stokes}}}{=} \int_{\partial D^3} i^*\omega \overset{\text{Stokes}}{=} \int_{D^3} d\, i^*\omega =$

D^3 palla di centro 0 e raggio 1

$= \mathrm{Volume}(D^3) \neq 0$

(c) $i^*\omega$ non è esatta. Infatti, se

$i^*\omega = d\alpha$

$\displaystyle\Rightarrow \int_{S^3} i^*\omega \overset{\text{stokes}}{=} \int_{\underbrace{\partial S^3}_{=\emptyset}} \alpha = 0$ ↯.