

Semiclassical theory of conduction in metals - The Boltzmann equation

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(reference: Ashcroft & Mermin, Solid State Physics, Ch 13 & 16)

1 Introduction - Electronic transport - generalities

Up to here, we have applied the semiclassical model of electron dynamics to the cases of :

- static electric fields
- uniform and static magnetic fields
- [uniform and static electric and magnetic crossed fields: not done, not requested for the exam; but see A&M]

From now on we will treat a **more general case** (presence of \mathbf{E} , \mathbf{H} , ∇T , functions of \mathbf{r} and t ; dependence of μ on \mathbf{r}) but always considering:

- Independent electrons
- semiclassical motion between collisions
- no interband transition (conservation of band index n)
- no spin change (conservation of spin).

2 Sources of electronic scattering (collisions)

Ch 16 §I

- Perfect periodic crystal \implies NO COLLISIONS
- INDEPENDENT ELECTRON PICTURE (here!); main source of collisions:
 1. point defects, impurities, vacancies
 2. thermal effects \implies vibrations (small!) of electrons around their equilibrium positions (amplitude of vibrations depending on T : important scattering source in DC and main responsible of the T dependence of conductivity around R.T.; as $T \rightarrow 0\text{K}$, defects dominate)
- BEYOND INDEPENDENT ELECTRON APPROXIMATION:
 e^-e^- scattering (e^-e^- interactions) that are however:
 - $\ll T$ -dependent lattice vibrations effects at high T ;
 - \ll impurity effects at low T

3 Non-equilibrium distribution function

A&M Ch. 13 §Introduction p 244 + Ch. 16 §IV p 319 + CH. 16 §I p 315

3.1 Generalities

We consider a **non equilibrium distribution function** $g(\mathbf{r}, \mathbf{k}, t)$ in phase space (in general, due to *applied fields* or *temperature gradients*), Is the occupancy of state \mathbf{k} at position \mathbf{r} and time dt .

External forces act to drive the distribution function **away** from equilibrium.

The limiting case at equilibrium, with NO applied fields, NO T gradients), corresponds to:

$$g^0(\mathbf{k}) = f(\mathcal{E}(\mathbf{k})) = \frac{1}{e^{(\mathcal{E}(\mathbf{k})-\mu)/k_B T} + 1} \quad (1)$$

AIM: derive a **closed** expression for $g(\mathbf{r}, \mathbf{k}, t)$ (when possible!) using:

- assumption of semiclassical equations of motion between collisions
- simple treatment of collisions

GENERALIZATION (the simplest possible!) OF THE RELAXATION TIME:

We continue to assume that an e^- experiences a collision in a time interval dt with probability dt/τ , but τ is in general $\tau(\mathbf{r}, \mathbf{k})$.

Rationale: even in the independent electron approximation, collisions are not totally random and uncorrelated. They depend on the distribution of all the e^- (for instance, on the occupation of the other states \mathbf{k}).

3.2 Differential equation for $g(\mathbf{r}, \mathbf{k}, t)$

We construct in general $g(\mathbf{r}, \mathbf{k}, t)$ at time t from its value at time $t' = t - dt$

- a) first assuming NO collision during the infinitesimal time interval dt
- b) consider \mathbf{r} and \mathbf{k} evolving according to the semiclassical equations of motion:

$$\dot{\mathbf{r}} = \mathbf{v}(\mathbf{k}) \quad (2)$$

$$\hbar \dot{\mathbf{k}} = -e(\mathbf{E} + \frac{1}{c} \mathbf{v} \times \mathbf{H}) = \mathbf{F}(\mathbf{r}, \mathbf{k}). \quad (3)$$

We consider explicitly the time evolution of \mathbf{r} and \mathbf{k} from t' to t (linear expansion in dt):

$$t' = t - dt \quad \longrightarrow t \quad (4)$$

$$\mathbf{r}' = \mathbf{r} - \mathbf{v}(\mathbf{k})dt \quad \longrightarrow \mathbf{r} \quad (5)$$

$$\mathbf{k}' = \mathbf{k} - \frac{\mathbf{F}}{\hbar}dt \quad \longrightarrow \mathbf{k} \quad (6)$$

The number of e^- that occupy the volume of phase space $\Delta \mathbf{r} \Delta \mathbf{k}$ centered in \mathbf{r} and \mathbf{k} at the time t is:

$$\frac{\Delta \mathbf{r} \Delta \mathbf{k}}{8\pi^3} g(\mathbf{r}, \mathbf{k}, t). \quad (7)$$

(we do not consider spin degeneracy, since we focus on a precise spin channel and we remain in that one). Analogously, the number of e^- occupying the volume of phase space

$\Delta\mathbf{r}'\Delta\mathbf{k}'$ centered in \mathbf{r}' and \mathbf{k}' at the time t' is:

$$\frac{\Delta\mathbf{r}'\Delta\mathbf{k}'}{8\pi^3}g(\mathbf{r}',\mathbf{k}',t') = \frac{\Delta\mathbf{r}'\Delta\mathbf{k}'}{8\pi^3}g\left(\mathbf{r} - \mathbf{v}(\mathbf{k})dt, \mathbf{k} - \frac{\mathbf{F}}{\hbar}dt, t - dt\right). \quad (8)$$

If there are **no collisions (no scattering)**, the trajectory of the e^- in phase space is such that all the e^- that are in the volume of phase space $\Delta\mathbf{r}'\Delta\mathbf{k}'$ centered in \mathbf{r}' and \mathbf{k}' at the time t' , are also in the volume of phase space $\Delta\mathbf{r}\Delta\mathbf{k}$ centered in \mathbf{r} and \mathbf{k} at the time t , therefore:

$$\frac{\Delta\mathbf{r}\Delta\mathbf{k}}{8\pi^3}g(\mathbf{r},\mathbf{k},t) = \frac{\Delta\mathbf{r}'\Delta\mathbf{k}'}{8\pi^3}g\left(\mathbf{r} - \mathbf{v}(\mathbf{k})dt, \mathbf{k} - \frac{\mathbf{F}}{\hbar}dt, t - dt\right). \quad (9)$$

For the Liouville theorem:

$$\frac{\Delta\mathbf{r}\Delta\mathbf{k}}{8\pi^3} = \frac{\Delta\mathbf{r}'\Delta\mathbf{k}'}{8\pi^3} \quad (10)$$

therefore in absence of scattering:

$$g(\mathbf{r},\mathbf{k},t) - g\left(\mathbf{r} - \mathbf{v}(\mathbf{k})dt, \mathbf{k} - \frac{\mathbf{F}}{\hbar}dt, t - dt\right) = 0. \quad (11)$$

Finally, we consider the expansion of $g\left(\mathbf{r} - \mathbf{v}(\mathbf{k})dt, \mathbf{k} - \frac{\mathbf{F}}{\hbar}dt, t - dt\right)$ in terms of $g(\mathbf{r},\mathbf{k},t)$ up to the linear term in dt and we get:

$$g\left(\mathbf{r} - \mathbf{v}(\mathbf{k})dt, \mathbf{k} - \frac{\mathbf{F}}{\hbar}dt, t - dt\right) = g(\mathbf{r},\mathbf{k},t) - \frac{\partial}{\partial\mathbf{r}}g \cdot \mathbf{v}(\mathbf{k})dt - \frac{\partial}{\partial\mathbf{k}}g \cdot \frac{\mathbf{F}}{\hbar}dt - \frac{\partial}{\partial t}g dt \quad (12)$$

therefore in absence of scattering we get:

$$\frac{\partial}{\partial\mathbf{r}}g \cdot \mathbf{v}(\mathbf{k}) + \frac{\partial}{\partial\mathbf{k}}g \cdot \frac{\mathbf{F}}{\hbar} + \frac{\partial}{\partial t}g = 0 \quad (13)$$

Introducing collisions, i.e., generalizing **in presence of scattering**, the right hand side of the previous equation is no longer zero, but it is the contribution due to scattered electrons:

$$\underbrace{\frac{\partial}{\partial\mathbf{r}}g \cdot \mathbf{v}(\mathbf{k}) + \frac{\partial}{\partial\mathbf{k}}g \cdot \frac{\mathbf{F}}{\hbar} + \frac{\partial}{\partial t}g}_{\text{DRIFT term}} = \underbrace{\left(\frac{\partial g}{\partial t}\right)}_{\text{COLLISION term}} \quad (14)$$

where the left side is the *DRIFT term* and the right side is the *COLLISION term*.

This is the BOLTZMANN EQUATION.

If we specify the forces and the collision term, we have an initial value problem to determine $g(\mathbf{r},\mathbf{k},t)$. The difficult part now is related to the explicit knowledge of the collision term, that in general could be very complicate.

The simplest possible approximation is the *relaxation time approximation* to replace the right hand side term with something much simpler, but this will come later on.

3.3 Change of $g(\mathbf{r}, \mathbf{k}, t)$ due to collisions

A&M CH. 16 §IV

We assume that the collisions are very well **localized in space and time**, so that those occurring to electrons in \mathbf{r} at time t are totally determined by the properties of the system in the neighborhoods of \mathbf{r} and close to time $t \implies$ for simplicity, we drop the explicit dependence of $g(\mathbf{r}, \mathbf{k}, t)$ on \mathbf{r} and t in what follows.

We consider therefore collisions that change instantaneously the crystal momentum from a volume $\Delta\mathbf{k}$ centered in \mathbf{k} to a volume $\Delta\mathbf{k}'$ centered in \mathbf{k}' . We distinguish scattering events that **increase** the occupancy of electronic states at \mathbf{k} and those that **decrease** it:

$$\left(\frac{\partial g}{\partial t}\right)_{coll}^{IN} > 0, \quad \left(\frac{\partial g}{\partial t}\right)_{coll}^{OUT} < 0. \quad (15)$$

We do not need to specify which is the scattering mechanism; it will be described in any case by some *scattering matrix* $W_{\mathbf{k}\mathbf{k}'}$ for one electron suffering a scattering from a state \mathbf{k} to \mathbf{k}' .

OUT: The e^- that are scattered OUT from the volume $\Delta\mathbf{k}$ centered in \mathbf{k} will go somewhere else in phase space, in volumes $\Delta\mathbf{k}'$ centered in all the possible \mathbf{k}' , provided that the states are available (not already occupied, because the Pauli principle must be satisfied). Therefore, the contribution of such scattering events to the variation of $g(\mathbf{r}, \mathbf{k}, t)$ is:

$$\left(\frac{\partial g}{\partial t}\right)_{coll}^{OUT} = -g(\mathbf{k}) \int \frac{\Delta\mathbf{k}'}{8\pi^3} W_{\mathbf{k}\mathbf{k}'} (1 - g(\mathbf{k}')) \quad (16)$$

where we have also introduced a weighting factor $g(\mathbf{k})$ which is the occupancy of the states in the volume $\Delta\mathbf{k}$ centered in \mathbf{k} and the sign $-$ since it is a reduction in the occupancy.

IN: The e^- that are scattered IN the volume $\Delta\mathbf{k}$ centered in \mathbf{k} come from somewhere else in phase space, from volumes $\Delta\mathbf{k}'$ centered in all the possible \mathbf{k}' , provided that those states are filled. Therefore, the contribution of such scattering events to the variation of $g(\mathbf{r}, \mathbf{k}, t)$ is:

$$\left(\frac{\partial g}{\partial t}\right)_{coll}^{IN} = (1 - g(\mathbf{k})) \int \frac{\Delta\mathbf{k}'}{8\pi^3} W_{\mathbf{k}'\mathbf{k}} g(\mathbf{k}') \quad (17)$$

where we have also introduced a weighting factor $(1-g(\mathbf{k}))$ which is the availability of the states in the volume $\Delta\mathbf{k}$ centered in \mathbf{k} (they should not be already occupied, because the Pauli principle must be satisfied).

Keep in mind that these scattering events are supposed to be LOCALIZED (depending only on \mathbf{r} and t , not on \mathbf{r}' and t'). The **total balance** of the scattering events gives:

$$\left(\frac{\partial g}{\partial t}\right)_{coll} = - \int \frac{\Delta \mathbf{k}'}{8\pi^3} \{W_{\mathbf{k}\mathbf{k}'}g(\mathbf{k}) [1 - g(\mathbf{k}')] - W_{\mathbf{k}'\mathbf{k}}g(\mathbf{k}') [1 - g(\mathbf{k})]\} \quad (18)$$

4 Relaxation time approximation

If we consider now that the scattering OUT events depend only locally on the distribution $g(\mathbf{k})$ around \mathbf{k} , and that the scattering IN events depend only on the *local equilibrium* distribution function $g^0(\mathbf{k})$ (local equilibrium *prior to* the collisions), we can dramatically simplify the IN and OUT collisions terms as follows:

$$\left(\frac{\partial g}{\partial t}\right)_{coll}^{OUT} = - \frac{g(\mathbf{k})}{\tau(\mathbf{k})} \quad (19)$$

$$\left(\frac{\partial g}{\partial t}\right)_{coll}^{IN} = \frac{g^0(\mathbf{k})}{\tau(\mathbf{k})} \quad (20)$$

where $\tau(\mathbf{k})$ is some *relaxation time*. The final result:

$$\left(\frac{\partial g}{\partial t}\right)_{coll} \approx - \frac{g(\mathbf{k}) - g^0(\mathbf{k})}{\tau(\mathbf{k})} \quad (21)$$

which gives the Boltzmann equation in this form:

$$\frac{\partial}{\partial t}g + \frac{\partial}{\partial \mathbf{r}}g \cdot \mathbf{v}(\mathbf{k}) + \frac{\partial}{\partial \mathbf{k}}g \cdot \frac{\mathbf{F}}{\hbar} = - \frac{\delta g}{\tau(\mathbf{k})} \quad (22)$$

suggests that collisions tend to restore the equilibrium, balancing the effect of the drift terms.

5 Scattering from isotropic materials

In case of *isotropic materials* and *stationary state* and keeping the relaxation time approximation, it is convenient to focus on the *variation* of $g(\mathbf{k})$, by defining:

$$\delta g(\mathbf{k}) = g(\mathbf{k}) - g^0(\mathbf{k}) \quad (23)$$

In this case the equilibrium distribution function $g^0(\mathbf{k})$ is dependent *neither on \mathbf{r} nor on t* , but only on \mathbf{k} . Therefore, since:

$$\frac{\partial \delta g}{\partial t} = \frac{\partial g}{\partial t} \quad (24)$$

$$\frac{\partial \delta g}{\partial \mathbf{r}} = \frac{\partial g}{\partial \mathbf{r}} \quad (25)$$

$$\frac{\partial \delta g}{\partial \mathbf{k}} = \frac{\partial (g - g^0)}{\partial \mathbf{k}} \quad (26)$$

we finally get that the Boltzmann equation reduces to:

$$\frac{\partial}{\partial t}\delta g + \frac{\partial}{\partial \mathbf{r}}\delta g \cdot \mathbf{v}(\mathbf{k}) + \frac{\partial}{\partial \mathbf{k}}(g^0 + \delta g) \cdot \frac{\mathbf{F}}{\hbar} = -\frac{\delta g}{\tau(\mathbf{k})} \quad (27)$$

6 Applications and examples

Remaining within the assumption of SMALL FIELDS, we will discuss the following cases:

- Isotropic perturbations \implies we can use Eq. 27
 - D.C., static and uniform \mathbf{E} (stationary state)
 - A.C., using Linear Response Theory
- Materials with $\nabla_{\mathbf{r}}T \neq 0$ and $\mu = \mu(\mathbf{r}) \implies$ we must use Eq. 22
 - D.C., static and uniform \mathbf{E} (stationary state)
 - $\nabla T = 0$ but $\mu = \mu(\mathbf{r})$; applied static and uniform \mathbf{E} and \mathbf{H}

6.1 Isotropic perturbation: static and uniform \mathbf{E} applied (stationary state)

In case of stationary state with static and uniform \mathbf{E} applied, Eq. 27 further simplifies, since $\frac{\partial}{\partial t}\delta g=0$ (stationary state) and $\frac{\partial}{\partial \mathbf{r}}\delta g=0$ (deviations from the equilibrium cannot depend on the spatial point). Furthermore, $\frac{\partial}{\partial \mathbf{k}}\delta g \cdot \mathbf{E}$ must be neglected, being infinitesimal of second order in \mathbf{E} , so that the gradient in \mathbf{k} contains only the contribution:

$$\frac{\partial}{\partial \mathbf{k}}g^0 = \nabla_{\mathbf{k}}\mathcal{E}(\mathbf{k})\frac{\partial g^0}{\partial \mathcal{E}} = \hbar\mathbf{v}(\mathbf{k})\frac{\partial g^0}{\partial \mathcal{E}} \quad (28)$$

Therefore we get:

$$\delta g = e\mathbf{E} \cdot \mathbf{v}(\mathbf{k})\tau(\mathbf{k})\frac{\partial g^0}{\partial \mathcal{E}} \quad (29)$$

The equation that relates current and electron velocity, which is $\mathbf{j} = -nev$ in the Drude model, becomes:

$$\mathbf{j} = -e \sum_{\mathbf{k}} \delta g(\mathbf{k})\mathbf{v}(\mathbf{k}) = -e^2 \int \frac{d\mathbf{k}}{(2\pi)^3} (\mathbf{E} \cdot \mathbf{v}(\mathbf{k})) \mathbf{v}(\mathbf{k})\tau(\mathbf{k})\frac{\partial g^0}{\partial \mathcal{E}} \quad (30)$$

We can see from the previous eq. the tensorial character of the conductivity:

$$j_i = \sum_j \sigma_{ij}E_j \quad (31)$$

with

$$\sigma_{ij} = e^2 \int \frac{d\mathbf{k}}{(2\pi)^3} \tau(\mathbf{k})v_i(\mathbf{k})v_j(\mathbf{k}) \left[-\frac{\partial g^0}{\partial \mathcal{E}(\mathbf{k})} \right] \quad (32)$$

Note:

- **Anisotropy:** in general \mathbf{j} is NOT parallel to \mathbf{E} , at variance with free electron case. In case of *cubic* materials, however, the expression of σ is simplified: $\sigma_{ij} = \delta_{ij}\sigma$.
- **Filled bands are inert:** only deviations from full filling are important
- **Importance of Fermi surfaces:** Although the integral seems to be done over the whole BZ, the term $\left[\frac{\partial g^0}{\partial \mathcal{E}(\mathbf{k})}\right]$ under the integral is non zero only around the Fermi surface (since it is basically $\propto \delta(\mathcal{E} - \mathcal{E}_F)$). The conductivity is determined by the conduction band in an interval of size $k_B T$ around \mathcal{E}_F .
- **Particular case: parabolic band and effective mass.** Considering that: (i) $-\frac{\partial g^0}{\partial \mathcal{E}(\mathbf{k})} = \delta(\mathcal{E} - \mathcal{E}_F)$; (ii) averaging over all the electrons, $\langle v_i v_j \rangle = \langle v_i^2 \rangle \delta_{ij} = \frac{1}{3} v^2 \delta_{ij}$; (iii) accounting for a spin factor of 2 if the two spin channels equally contribute to the current; we have:

$$\begin{aligned}
 \sigma &= 2e^2 \int \frac{d\mathbf{k}}{(2\pi)^3} \tau(\mathbf{k}) \frac{1}{3} v^2(\mathbf{k}) \delta(\mathcal{E} - \mathcal{E}_F) \\
 &= \frac{e^2}{12\pi^3} \int_{Fermi\ surf} \tau(\mathbf{k}) v^2(\mathbf{k}) \frac{dS}{|\nabla_{\mathbf{k}} \mathcal{E}(\mathbf{k})|} \\
 &= \frac{e^2}{12\pi^3} \int_{Fermi\ surf} \tau(\mathbf{k}) v(\mathbf{k}) \frac{1}{\hbar} dS
 \end{aligned} \tag{33}$$

Furthermore, considering that in case of parabolic and isotropic band, the Fermi surface is a Fermi sphere with radius k_F and the integral of dS over such surface simply gives the surface area, we finally get:

$$\sigma = \frac{e^2}{12\pi^3} \tau_F v_F \frac{1}{\hbar} 4\pi k_F^2 = \frac{ne^2 \tau_F}{m^*} \tag{34}$$

which resembles the well known result for free electrons, BUT with $m \rightarrow m^*$ and $\tau \rightarrow \tau_F$. Attention: in the final result, the fact that ONLY the electrons around \mathcal{E}_F do contribute to the current is hidden, but it is clear by following the derivation!

_____END OF THE EXAMPLES DISCUSSED IN CLASS IN 2017/18 _____

6.2 Isotropic perturbation:

A.C. conductivity, using Linear Response Theory

A&M, Ch 13, p. 252

We start again from the linearized eq. 27 for δg , but since now \mathbf{E} is time dependent, we must keep all the terms in $\frac{\partial}{\partial t}$. As in the previous case, $\frac{\partial}{\partial \mathbf{k}} \delta g \cdot \mathbf{E}$ must be neglected, being infinitesimal of second order in \mathbf{E} , so that we get:

$$\frac{\partial}{\partial t} \delta g + \frac{\partial}{\partial \mathbf{r}} \delta g \cdot \mathbf{v}(\mathbf{k}) - \frac{e\mathbf{E}}{\hbar} \cdot \frac{\partial g^0}{\partial \mathbf{k}} = -\frac{\delta g}{\tau(\mathbf{k})} \quad (35)$$

Consider $\mathbf{E}(\mathbf{r}, t) = \mathbf{E}_0 e^{i(\mathbf{q}\cdot\mathbf{r} - \omega t)}$. In case of linear response regime, we have: $\delta g(\mathbf{r}, \mathbf{k}, t) = \Phi(\mathbf{k}) e^{i(\mathbf{q}\cdot\mathbf{r} - \omega t)}$. Substituting in eq. 35, we get:

$$-i\omega\Phi + \mathbf{v} \cdot i\mathbf{q}\Phi - \frac{e}{\hbar} \mathbf{E}_0 \cdot \frac{\partial g^0}{\partial \mathbf{k}} = -\frac{\Phi}{\tau} \quad (36)$$

from which, using eq. 28, we obtain:

$$\Phi(\mathbf{k}) = \frac{e\tau \mathbf{E}_0 \cdot \mathbf{v}}{1 - i\tau(\omega - \mathbf{q} \cdot \mathbf{v})} \frac{\partial g^0}{\partial \mathcal{E}} \quad (37)$$

Considering the expression for the current in terms of the electron velocity (always remind the analogous in the Drude model, $\mathbf{j} = -ne\mathbf{v}$):

$$\begin{aligned} \mathbf{j} &= -e \int \frac{d\mathbf{k}}{(2\pi)^3} \delta g(\mathbf{k}) \mathbf{v}(\mathbf{k}) \\ &= -e \int \frac{d\mathbf{k}}{(2\pi)^3} \Phi(\mathbf{k}) e^{i(\mathbf{q}\cdot\mathbf{r} - \omega t)} \mathbf{v}(\mathbf{k}) \\ &= -e \int \frac{d\mathbf{k}}{(2\pi)^3} \frac{e\tau \mathbf{E} \cdot \mathbf{v}}{1 - i\tau(\omega - \mathbf{q} \cdot \mathbf{v})} \frac{\partial g^0}{\partial \mathcal{E}} \mathbf{v}(\mathbf{k}) \end{aligned} \quad (38)$$

and the expression for the current in terms of conductivity and electric field (eq. 31), we get:

$$\sigma_{ij} = e^2 \int \frac{d\mathbf{k}}{(2\pi)^3} \frac{\tau(\mathbf{k}) v_i(\mathbf{k}) v_j(\mathbf{k})}{1 - i\tau(\mathbf{k})(\omega - \mathbf{q} \cdot \mathbf{v}(\mathbf{k}))} \left[-\frac{\partial g^0}{\partial \mathcal{E}(\mathbf{k})} \right] \quad (39)$$

which reduces to the static case previously discussed (eq. 32) when $\mathbf{q} \rightarrow 0$ (long wavelengths, field uniform in space) and $\omega \rightarrow 0$ (static limit).

6.3 Anisotropic material ($\nabla T \neq 0$, $\mu = \mu(\mathbf{r})$) static and uniform \mathbf{E} (stationary state)

In this case also g^0 depends on \mathbf{r} , so that we must use eq. 22. We have:

$$\frac{\partial g}{\partial \mathbf{r}} = \frac{\partial g}{\partial \mu} \cdot \nabla \mu + \frac{\partial g}{\partial T} \cdot \nabla T \quad (40)$$

and since, being consistent with the linear expansion in the perturbative terms:

$$\begin{aligned} \frac{\partial g}{\partial \mu} &\approx \frac{\partial g^0}{\partial \mu} = \frac{g^0}{k_B T} e^{\frac{\mathcal{E} - \mu}{k_B T}} \\ \frac{\partial g}{\partial T} &\approx \frac{\partial g^0}{\partial T} = \frac{\mathcal{E} - \mu}{(k_B T)^2} (g^0)^2 k_B e^{\frac{\mathcal{E} - \mu}{k_B T}} \end{aligned} \quad (41)$$

Then, since $\frac{\partial g}{\partial \mathcal{E}} \approx \frac{\partial g^0}{\partial \mathcal{E}} = -\frac{(g^0)^2}{k_B T} e^{\frac{\mathcal{E}-\mu}{k_B T}}$, we can express the right-hand side terms of eq. 41 in terms of $\frac{\partial g^0}{\partial \mathcal{E}}$, obtaining:

$$\begin{aligned}\frac{\partial g}{\partial \mu} &\approx -\frac{\partial g^0}{\partial \mathcal{E}} \\ \frac{\partial g}{\partial T} &\approx -\frac{\mathcal{E}-\mu}{T} \frac{\partial g^0}{\partial \mathcal{E}}\end{aligned}\quad (42)$$

Similarly, also $\frac{\partial g}{\partial \mathbf{k}}$ can be expressed in terms of $\frac{\partial g^0}{\partial \mathcal{E}}$, obtaining:

$$\frac{\partial g}{\partial \mathbf{k}} = \frac{\partial g}{\partial \mathcal{E}} \frac{\partial \mathcal{E}}{\partial \mathbf{k}} \approx \frac{\partial g^0}{\partial \mathcal{E}} \frac{\partial \mathcal{E}}{\partial \mathbf{k}} = \frac{\partial g^0}{\partial \mathcal{E}} \hbar \mathbf{v}(\mathbf{k}). \quad (43)$$

Substituting in eq. 22 and considering the stationary regime $\left(\frac{\partial g}{\partial t} = 0\right)$, we get, a part from terms of higher order in $\nabla_{\mathbf{r}} T$ and \mathbf{E} :

$$\tau(\mathcal{E}(\mathbf{k})) \mathbf{v}(\mathbf{k}) \cdot \left(\nabla_{\mathbf{r}} \mu + \frac{\mathcal{E}-\mu}{T} \nabla_{\mathbf{r}} T + e\mathbf{E} \right) \left(-\frac{\partial g^0}{\partial \mathcal{E}} \right) = g^0(\mathbf{k}) - g(\mathbf{k}) \quad (44)$$

which is eq. (13.43) of A&M book (Ch. 13.2).

6.4 Anisotropic material ($\nabla T = 0$ but $\mu = \mu(\mathbf{r})$) static and uniform \mathbf{E} and \mathbf{H} (stationary state)

For the sake of simplicity, we consider now $\nabla T = 0$. We can start from the last eq. 44. The external force term, $e\mathbf{E}$, must be substituted with $e\mathbf{E} + \frac{1}{c} \mathbf{v} \times \mathbf{H}$. It is interesting to notice that $\mathbf{v} \cdot \mathbf{v} \times \mathbf{H} = 0$, so that the external applied magnetic field has NO EFFECT! Furthermore, if we consider the case of a parabolic and isotropic band, so that $\mathcal{E}(\mathbf{k}) = \frac{\hbar^2 k^2}{2m^*}$, we have that $\mathbf{v} = \frac{\hbar \mathbf{k}}{m^*}$ (note that only in this approximation \mathbf{v} is parallel to \mathbf{k} !), and eq. 44 becomes:

$$\underbrace{\tau(\mathcal{E}(\mathbf{k})) \left(-\frac{\partial g^0}{\partial \mathcal{E}} \right) (\nabla_{\mathbf{r}} \mu + e\mathbf{E}) \frac{\hbar}{m^*} \cdot \mathbf{k}}_{\text{dependent only on } \mathcal{E}} = g^0(\mathbf{k}) - g(\mathbf{k}). \quad (45)$$

The part $\underbrace{\hspace{10em}}$ in the left-hand side term is dependent only on \mathcal{E} , so that we can rewrite the last equation in a very simplified form:

$$\mathbf{a}(\mathcal{E}) \cdot \mathbf{k} = g^0(\mathbf{k}) - g(\mathbf{k}). \quad (46)$$

which is eq. (16.26) of A&M book.