PhD – Environmental Fluid Mechanics – Physics of the Atmosphere
University of Trieste – International Center for Theoretical Physics

Moisture and Temperatures in the Atmosphere

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Abstract

The first part of these notes is devoted to the way in which the amount of water vapour is quantified, then measured in atmospheric physics. The concept of humidity is presented and the different definitions of humidity are discussed. The second part of the lecture deals with the various kind of temperatures adopted in the physics of the atmosphere. Temperature are easy to measure quantities and deeply related to the energy transfers, for this reason several kids of temperatures are defined according to the different processes (transformations) acting on the atmospheric elementary volumes (parcels). Many of these temperatures are related to the amount of atmospheric water vapor, then are essentially ways to measure humidity.

Measuring moisture in atmospheric sciences

The dependence of saturation vapor pressure by absolute temperature alone helps us in measuring the amount of water vapor in atmosphere. Several quantities used to quantify moisture were developed, some of them with similar (or even equal names). One of these is *absolute humidity* which is defined as the water vapor density

$$Ah \equiv \rho = \frac{e}{RT}$$

This quantity is useful when dealing with processes that are strictly related to the bulk amount of water vapor in a fixed volume, e.g., absorption of radiation. This moisture measurement is seldom used for meteorological purposes and in its place another quantity called *specific humidity* is used. The specific humidity is the ratio between the current water vapor density in a fixed volume (parcel) and the total parcel density, that is

$$q \equiv \frac{\rho_v}{(\rho)} = \frac{\rho_v}{(\rho_v + \rho_d)} = \frac{e}{(R_v T)} \frac{(R_{air} T)}{p} = \frac{R_{air}}{R_v} \frac{e}{p} \simeq 0.622 \frac{e}{p}$$

where we used the ideal gas law for the air and water vapor to express specific humidity as a function pressures. Remember that p is the total air pressure (i.e., that measured by the barometer).

Another quantity often used in meteorology is mixing ratio defined as the ratio between the water vapour density in a parcel and the dry air density (i.e., withdrawing the contribution of water vapour to total density). With this definition we obtain

$$r \equiv \frac{\rho_v}{(\rho_{dry})} = \frac{\rho_v}{(\rho_{air}-\rho_v)} = \frac{e}{(R_v T)} \frac{1}{\left(\frac{p}{(R_{air}T)} - \frac{e}{(R_v T)}\right)} = \frac{R_{air}}{R_v}\frac{e}{(p-\frac{R_{air}}{R_v}e)} \simeq 0.622\frac{e}{(p-0.622e)} \sim 0.622\frac{e}{p}$$

This means that for almost all the meteorological purposes mixing ratio and specific humidity can be considered almost the same (substitute *p=1013 hPa* and *e =6.11 hPa* in the above formula for a check).

Both the above definitions, even if widely used in atmospheric physics, have two slight disadvantages: the first is that both depend upon air pressure (the same amount of water vapour molecules has a different specific humidity and mixing ratio for different air pressures); the second is that both specific humidity and mixing ratio do not give any information concerning where are we in respect to saturation.

The first disadvantage is more theoretical than practical (meteorologists are very rude people). In fact for the great majority of purposes pressure, during horizontal (equipotential, indeed) displacements, just by a few tens of hPa. Try to substitute *1013 hPa* with *993 hPa* or *1033 hPa* to have an idea of pressure influence on specific humidity and mixing ratio.

The second disadvantage is a little bit more tricky. In fact for the great majority of situations, water vapour is negligible apart when condensation takes place. In that case even if the amount of water vapour is small compared to the remaining atmospheric constituents, because of its latent heat of condensation, its presence becomes fundamental for atmospheric (thermo)dynamics.

For this reason meteorologists invented another quantity to quantify moisture in atmosphere, which is *relative humidity*. Relative humidity is defined as the ratio between the observed water vapour amount and the saturation for the observed parcel's fixed temperature. The definition seems straightforward but conceptually it depends by the way in which you define the “water vapour amount”. Two definitions are available and are used, the first recommended by WMO uses the *mixing ratio* r, the second used in the great majority of applications and textbooks uses the *specific humidity* q. The two definitions of relative humidity follow here below

$$r_h = \frac{r}{r_s} = \frac{0.622e}{(p-0.622e)}\frac{(p-0.622e_s)}{0.622e_s} = \frac{e}{e_s}\frac{(p-0.622e_s)}{(p-0.622e)}$$

$$r_h = \frac{q}{q_s} = \frac{0.622e}{p}\frac{p}{0.622e_s} = \frac{e}{e_s}$$

The fact we used the same acronym is not due to a typo but that is exactly what is currently done in all the theoretical and operational activities. In other words you cannot be sure what kind of definition of relative humidity is adopted apart when explicitly written.

In this case Nature helps mankind because for almost all the meteorological purposes the two definitions give almost the same result. The reason why textbooks recommend specific humidity as a moisture quantifier is that with specific humidity relative humidity does not depend upon pressure (neither slightly).

Relative humidity, often represented in percent values, gives you all the fundamental information you need concerning how far we are from saturation, but has the great disadvantage to be temperature dependent, in fact saturation specific humidity (or saturation mixing ratio) depends from temperature. For this reason another moisture measurement is often used that is the dew point temperature.

Temperatures in atmosphere

Temperature is a fundamental variable for the description and prediction of atmospheric processes because it gives information concerning the direction in which the exchanges of thermal energy take place. But the temperature of an air-mass or better of an air parcel can change accordingly to its evolution, it is useful to introduce several "temperatures" that are not real in the sense that they can be currently measured by a simple thermometer, but they are going to be reached when an air parcel is subject to a well defined transformation. For this reason these temperatures are function of the current air parcel properties (e.g. fraction of water vapor, ...) and of the transformation (e.g., isobaric, adiabatic, ...).

Dew point temperature

Dew temperature is defined as the temperature reached by an air-mass when it is cooled isobarically, keeping constant its content of water vapor (e.g, $de/dt=0$), up to the level in which the condensation of water vapor takes place. This temperature can be computed using the Clausius-Clapeyron equation

$$\frac{de_s}{dT}=\frac{1}{T}\frac{l_v}{v_v-v_w}$$

where e_s is the saturation vapor pressure, T the absolute temperature, l_v the latent heat of condensation per unit mass and v_v and v_w are respectively the specific volume (inverse of density) of water vapor and liquid water.

The above equation can be simplified, for atmospheric purposes, keeping in mind that the specific volume of water is usually very small compared to the specific volume of vapour (i.e., the density of liquid water is usually higher than water vapor density) then we can write

$$\frac{de_s}{dT}\approx\frac{l_V}{T\,v_V}$$

approximating the behavior of water vapor with an ideal gas we can write $e_s=\rho_v R_v T$ that, when inserted into the above equation, becomes

$$\frac{de_s}{dT}\approx\frac{l_V\,e_s}{R_V T^2}$$

This equation, when integrated, gives the following result

$$\frac{e_s}{e_{ref}}\approx\frac{e^{\frac{l_V}{R_V T_{ref}}}}{e^{\frac{l_V}{R_V T}}}$$

that can be written in the following, more readable, form

$$e_s\approx A e^{-\frac{l_V}{R_V T}}$$

where $A=e_{ref}\exp(\frac{l_V}{R_V T_{ref}})$ is a constant that can be experimentally determined for a reference temperature T_{ref}. This constant depends, once the reference temperature is fixed, depends only from the latent heat of vaporization and from the water vapor "gas constant". Latent heat of vaporization is not strictly constant, but it just slightly decreases with the increasing of temperature (less than 1% for the meteorological temperature range, $l_V \simeq 2.5\cdot 10^6\, J\, kg^{-1}$ at 0 °C and $l_V \simeq 2.3\cdot 10^6\, J\, kg^{-1}$ at 100 °C), for this reason it can be considered as constant. The above equation states that saturation water vapor pressure (over liquid water) is function of temperature only and, once inverted, it can be used to determine the dew point temperature of an air-mass, then

$$T_{dew} \approx \frac{l_V}{R_V \ln(\frac{A}{e})}$$

Dew point temperature does not depend from total dry air pressure p_d or from other variables than its partial vapor pressure e, for this reason it is a good marker of air-masses. In particular dew point temperature does not change if we modify its temperature only, then it can characterize an air-mass in spite of its diabatic heating due to, as an example, the diurnal cycle.

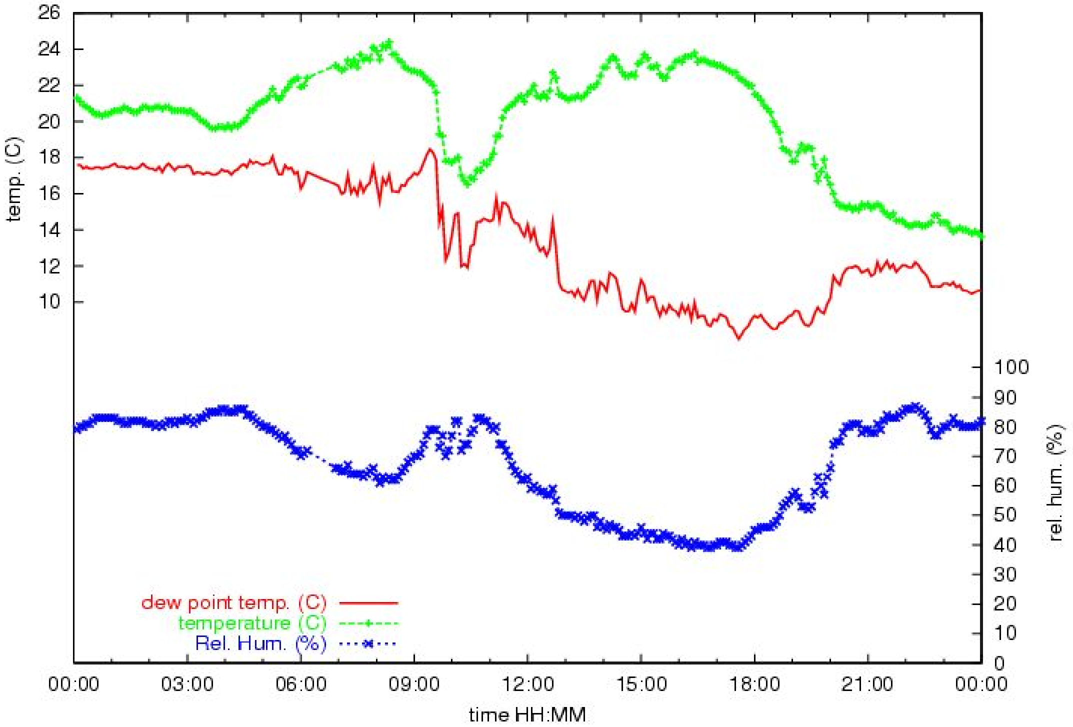

Fig.0. Dew point temperature (solid line), temperature (dashed line with crosses) and relative humidity (dotted line with) as measured by the Vivaro weather station (Friuli Venezia Giulia) during a tornado event that took place in the day of 4th June 1999 at 09:20 UTC.

It can be useful, for didactic purposes, to express dew point temperature as a function of relative humidity. In fact, once the air-mass relative humidity $rh=\frac{e}{e_s}$ is known as well as its absolute temperature, using the saturation water vapor pressure equation we can obtain its current partial vapor pressure e, then according to the isobaric transformation that defines dew point temperature, e will stay constant up to the moment in which saturation occurs. For an air-mass characterized by the relative humidity rh (expressed in relative values) and the absolute temperature T, the dew point temperature T_{dew} becomes

$$T_{dew}=\frac{T}{1-\frac{R_v T}{l_V}\ln(rh)}$$

Since $l_v \gg R_V T$ the above equation states that dew point temperature is always lower than the observed thermometer temperature and, for a fixed temperature, it increases with the increasing of relative humidity, coinciding with the observed thermometer temperature at saturation, that is when relative humidity $rh=1$.

Before to conclude the part related to the dew point, it is important to compute how dew point temperature changes with height. This can be easily done simply taking the z derivative of the equation that gives T_{dew} as a function of vapor pressure e. This differentiation gives

$$\frac{d}{dz}(T_{dew})=\frac{R_v}{l_V}\frac{T_{dew}^2}{e}\frac{d}{dz}(e)$$

Since e is proportional to the dry air pressure p_d by way of the relationship

$$e=\frac{R_d}{R_v} r\, p_d$$

obtained through the definition of mixing ratio $r \equiv e/p_d \cdot R_d/R_v$, the above derivative becomes

$$\frac{d}{dz}(T_{dew})=\frac{R_v}{l_V}\frac{T_{dew}^2}{p_d}\frac{d}{dz}(p_d)$$

then, using the equation of state for dry air ρ, we obtain

$$\frac{d}{dz}(T_{dew})=\frac{R_v}{l_V}\frac{T_{dew}^2}{R_d T}\frac{1}{\rho_d}\frac{d}{dz}(p_d)$$

At this point, assuming the hydrostatic approximation $\frac{1}{\rho_d}\frac{d}{dz}(p_d)=-g$, we can significantly simplify the above equation, that is

$$\frac{d}{dz}(T_{dew})=-\frac{R_v g}{R_d l_V}\frac{T_{dew}^2}{T}$$

remembering that the dry adiabatic lapse rate is given by the relation $\Gamma_d=-g/c_p^d$, where c_d^p is the specific heat at constant pressure for dry air, we can easily show that

$$\Gamma_{dew} = \frac{d}{dz}(T_{dew}) = \frac{R_v c_p^d}{R_d l_V} \frac{T_{dew}^2}{T} \Gamma_d$$

Substituting the constants and assuming that $T_{dew} \approx 270$ and $T \approx 280$ (the current values, being in the absolute scale, are not extremely important) we will have $\Gamma_{dew} \approx 0.2 \Gamma_d$. This simple equation gives the explanation of the cloud formation. In fact that relationship states that the thermal lapse rate of dry air is greater (in absolute value) than the dew point lapse rate. This means that, during the adiabatic ascent (cooling), the temperature of dry air decreases more than its dew point temperature. This is the reason why a lifted parcel of air might reach saturation. There are some other vapors, in any case not relevant under the meteorological point of view, where the dew point lapse rate is greater, in absolute value, than that of dry air. For those vapors clouds does not for because of the lifting but because of sinking.

Wet bulb temperature

Wet bulb temperature is defined as the temperature reached by air when liquid water ievaporates in it isobarically (e.g., at constant pressure) up to saturation. It is necessary not to confuse wet bulb temperature with dew point temperature. In fact dew point temperature is reached after an isobaric transformation (i.e., vapor pressure *e* is kept constant) in which temperature *T* decreases and the mixing ratio *r* is kept constant at the same time up to the temperature in which condensation occurs, that is

$$\frac{de}{dt} = 0 \ , \ \frac{dT}{dt} < 0 \ \text{and} \ \frac{dr}{dt} = 0 \ .$$

On the contrary wet bulb temperature is reached after an isobaric transformation of the air in which mixing ratio is increased by evaporation of water and, then, at the same time temperature decreases up to the level in which the saturation equilibrium is reached, that is

$$\frac{de}{dt} = 0 \ , \ \frac{dT}{dt} < 0 \ \text{and} \ \frac{dr}{dt} > 0 \ .$$

Even if this transformation is quite simple to describe, it is not so easy to practically realize it. One possibility, that is exactly what is operationally done, is that of producing a steady flow of the air we are interested in, over a source of moisture (water). If the flow is enough strong, after a certain amount of time that depends from the mass and exposed surface of the moisture source, a steady state is reached in which the temperature of the moisture source is no more decreasing. In this situation, the amount of thermal energy given by the flow to the moisture source is exactly the same amount of energy taken by the evaporation of the moisture source. In fact the flow, at this regime, does not give anymore thermal energy to the moisture source, otherwise the moisture source temperature had to increase. At the same time, the source of moisture is not giving anymore thermal energy to the air flow, otherwise the air flow temperature had to decrease. This is possible only when the outgoing flow of air is saturated and when its temperature is the same of the moisture source, otherwise we could not observed the equilibrium.

The so called wet bulb equation (or psychrometric equation) describes the asymptotically reached dynamic equilibrium, and it is represented by the following simple relationship

$$\Delta Q_{given\, by\, air} = \Delta Q_{absorbed\ by\ evaporation}$$

which is based on the general principle of energy conservation. The first term of the above equation can be split into the two components of energy given by the incoming dry air and by water vapor, that is

$$\Delta Q_d + \Delta Q_v = \Delta Q_{absorbed\ by\, evaporation}$$

the second term can be written keeping into account the mass of liquid water transformed in vapor ΔM_v and the latent heat per unit mass L_v, that is

$$\Delta Q_d + \Delta Q_v = \Delta M_v L_v$$

at this point the thermal energy given by dry air and water vapor can be written using the mass of dry air M_d and the mass of water vapor M_v

$$M_d c_p \Delta T + M_v c_p^v \Delta T = \Delta M_v L_v$$

then, dividing by M_d and remembering that water vapor mixing ratio is defined as

$$r = \frac{M_v}{M_d}$$

the above equation becomes

$$(c_p + r c_p^v) \Delta T = L_v \Delta r .$$

If we write as T_w and r_s respectively the equilibrium temperature and the saturation mixing ratio reached by the outgoing flow while T and r represent the actual temperature and mixing ratio of the incoming air, the above equation becomes

$$(c_p + r c_p^v)(T - T_w) = L_v (r_s - r)$$

where T_w is the wet bulb temperature of the incoming air.

Wet bulb temperature is an extremely useful quantity in the atmosphere sciences for several reasons. First of all it can be used to measure the amount of moisture of the air in fact, knowing the wet bulb temperature and the current temperature, we can retrieve the current air mixing ratio (remember that the saturation mixing ratio r_s can be determined as a function of the wet bulb temperature and pressure). Moreover, wet bulb temperature is the asymptotic value of the temperature reached by hydrometeors during their fall into the atmosphere, which is then useful for the determination of their melting and/or evaporating rate as well as for their impact on downdraft enhancement. Wet bulb temperature is even extremely useful in thermodynamic diagrams, because it is connected with another temperature, that is equivalent potential temperature, which is directly linked to the atmospheric potential instability.
The differences between temperature and wet bulb temperature are shown in figure 1, where these two variables are displayed in conjunction with the relative humidity.

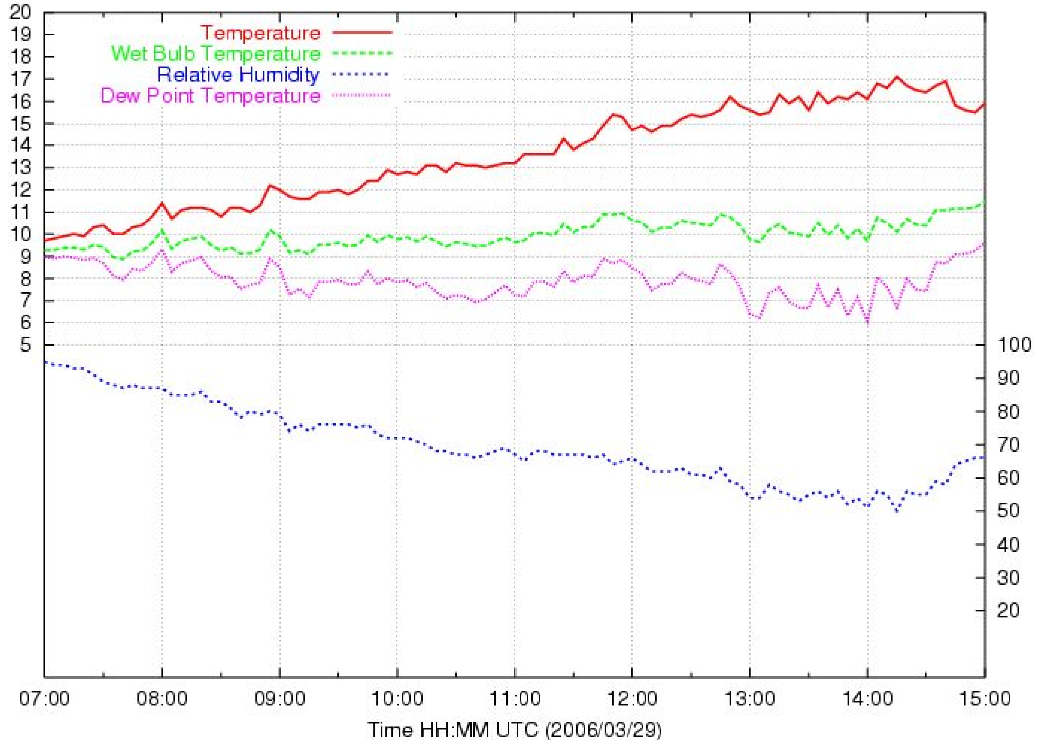

Figure 1. Temperature, wet bulb temperature, dew point temperature and relative humidity as observed at Udine between 07:00 and 15:00 UTC (solar time – 2 hours) an 29th March 2006.

In figure 1 it is clear that temperature and wet bulb temperature differ quite a lot, specially when atmosphere is far from saturation, as evidenced by the relative humidity trend (lowest panel). This is as well an explanation of why we can keep a bottle of drink cooler than its surroundings just maintaining it constantly wet. Moreover it is an evidence of how smart had been Nature in using sweating to keep cool the human bodies.

Wet bulb temperature depends form the mixing ratios r_s and r which, in turn, depend from pressure; for this reason wet bulb temperature depends from dry pressure as shown in figure 2. In that figure it is clear that the increasing of pressure produces a growth of wet bulb temperature. This can be naively interpreted keeping in mind that increasing of pressure, keeping constant the water vapor pressure and the saturation vapor pressure (which depends only from temperature), we cause a decrease in the mixing ratio, which is defined as the ratio between the mass of water vapor per mass of dry air. Increasing pressure, then, we increase the mass of dry air, then we increase the amount of thermal energy given by the air to the source of moisture. The differences in wet bulb temperature due to dry air pressure, in any case, are not particularly huge, at least for the typical atmospheric conditions and, obviously, disappear at saturation, when temperature, dew point temperature and wet bulb temperature coincide.

Operationally, wet bulb temperature is used to compute the mixing ratio of the air masses. This is done by way of two thermometers, one with a bulb kept wet and the other with the bulb kept dry. Both these thermometers are maintained under a vigorous steady vigorous (of the order of several meters per second). The difference observed in the two thermometers, when equilibrium is reached, is used to compute the air mass mixing ratio, then the relative humidity. Before to conclude it is important to specify that the process which bring to the wet bulb temperature, is not isentropic for the air mass, in fact this process is not adiabatic since thermal energy is withdrawn from the air mass.

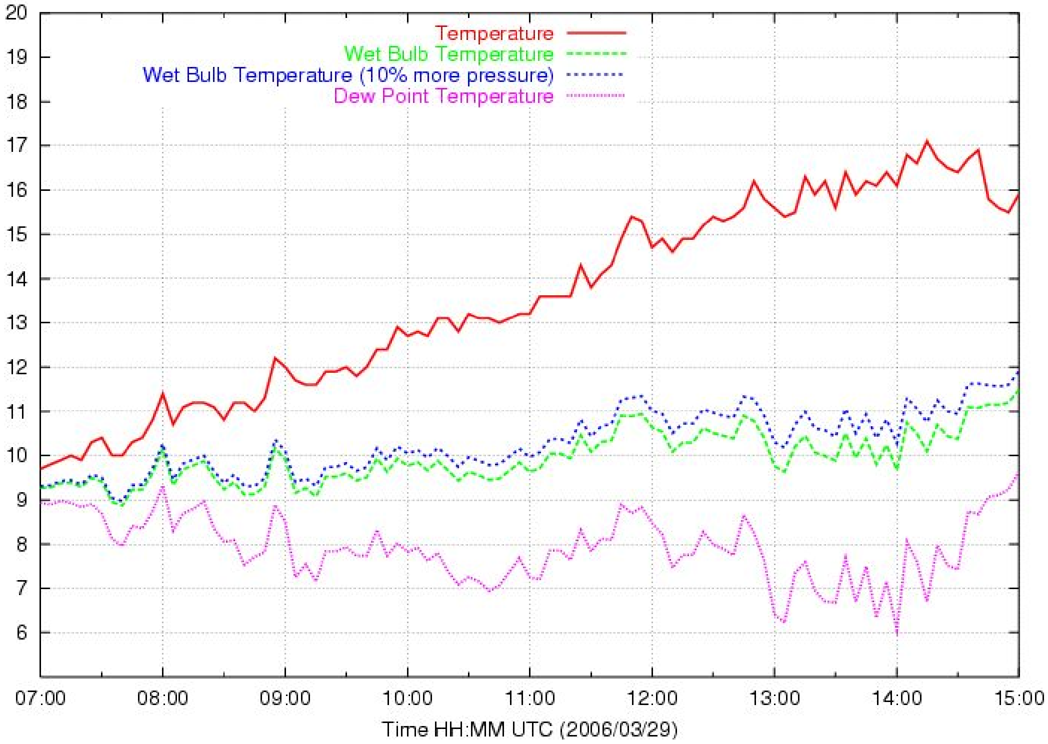

Figure 2. Temperature, wet bulb temperature for the observed pressure, wet bulb temperature for a 10% increased pressure and dew point temperature as observed at Udine between 07:00 and 15:00 UTC (solar time – 2 hours) an 29th March 2006.

Equivalent potential temperature

Equivalent potential temperature is the temperature reached by an air parcel (mixture of dry air and water vapor) when adiabatically lifted (i.e., cooled) up to a pressure such that all the water vapor is condensed and then brought back in a dry adiabatic descent (then after removing all its water content) up to a standard level pressure (usually 1000 hPa). This temperature is considered important because this process mimic the lifting of an air mass due to convection and it is useful to determine the potential and conditional instability. Equivalent potential temperature is not a real adiabatic process because water is fully removed before to start the dry adiabatic descent. Even if not completely correct, this temperature is nevertheless called equivalent potential and adiabatic. The expression pseudo-equivalent potential temperature is reserved to the temperature reached by an air mass which is lifted adiabatically, removing the condensed water as soon as it is formed, up to a pressure when all the water vapor is condensed and then draw back up to the standard level pressure with a dry (now all the water is disappeared) adiabatic descent. This temperature is called pseudo-equivalent potential temperature because it is not a pure adiabatic process even during the ascent of the air parcel in the sense that there is an exchange of matter with the environment. Pseudo-equivalent potential temperature mimic a process where the water condensed rains out as soon as it is formed, while equivalent potential temperature mimic a process where the condensed water rains out only from the top of the cloud. Reality is something in between, then both equivalent potential and pseudo-equivalent potential temperatures are only a model of real air ascents and they can differ by a few degrees.

The analytical expression of the equivalent potential temperature is obtained making use of the first principle of thermodynamics, the Clausius-Clapeyron equation and the rate of change of latent heat of vaporization with temperature. In particular, assuming that the process is adiabatic and reversible (with the caveats above mentioned) entropy must conserve, then

$$ds=\frac{dQ}{T}=0$$

where s is the entropy and dQ the global variation of thermal energy. When splitting dQ into the contributions due to the condensation of water vapor dQ_c , to the liquid water absorption dQ_w , the vapor absorption dQ_v , and the dry air absorption dQ_d , we will have

$$ds=\frac{dQ_c}{T}+\frac{dQ_w}{T}+\frac{dQ_v}{T}+\frac{dQ_d}{T}=0$$

Before to go ahead it is important to stress the fact that the above equation assumes that entropy is conserved (variation assumed as 0) without keeping into account the transition from water to ice, in other words the terms dQ_i and dQ_s due to the ice presence and solidification of liquid water are considered as null. This means that the equivalent potential temperature will be a conservative variable only if the ice formation is absent or negligible. In other words equivalent potential temperature computed for a real air mass might change because the liquid water is partially or totally removed during the lifting or because ice is forming in it.

The above equation can be expanded by way of the first principle of thermodynamics, in particular:

$$dQ_v=M_v(c_v^v dT+e\,dV)$$

where M_v is the vapor mass, c_v^v is the specific heat at constant volume for the vapor and *e* its partial vapor pressure;

$$dQ_d = M_d\,(c_p^d dT + p\,dV)$$

where M_d is the dry air mass, c_v^d is the specific heat at constant volume for the dry air and *p* its partial pressure;

$$dQ_w = M_w c^w dT$$

where M_w is the vapor mass, c^w is the water specific heat;

$$dQ_c = l_v\,dM_v$$

where dM_v is the mass of vapor that condenses and l_v is the latent heat of condensation.

Merging all these equations we obtain

$$l_v\frac{dM_v}{T} + M_w c^w\frac{dT}{T} + M_v(c_v^v dT + e\,dV)\frac{1}{T} + M_d(c_v^d dT + p\,dV)\frac{1}{T} = 0$$

assuming that both dry air and water vapor behave as ideal gases, we can use the equation of state for ideal gases,that is $e=\frac{M_v}{V}R_v T$ and $p=\frac{M_d}{V}R_d T$. These equations create a link between pressure and volume by way of temperature and the vapor and dry air constants (i.e, respectively R_v and R_d) . This link can be used to evaluate the volume differential in the above equation that, for unit mass, become respectively $dV=\frac{R_v}{e}dT-\frac{R_v}{e^2}de$ and $dV=\frac{R_d}{e}dT-\frac{R_d}{p^2}dp$.

Introducing these two expressions into the above entropy conservation equation we obtain

$$l_v\frac{dM_v}{T} + M_w c^w\frac{dT}{T} + M_v(c_v^v dT + e[\frac{R_v}{e}dT - \frac{R_v}{e^2}de])\frac{1}{T} + M_d(c_v^d dT + p[\frac{R_d}{p}dT - \frac{R_d}{p^2}dp])\frac{1}{T} = 0$$

then

$$l_v\frac{dM_v}{T} + M_w c^w\frac{dT}{T} + M_v(c_v^v dT + R_v dT - \frac{R_v}{e}de)\frac{1}{T} + M_d(c_v^d dT + R_d dT - \frac{R_d}{p}dp)\frac{1}{T} = 0\ .$$

Remembering that $R_v = c_p^d - c_v^d$ and $R_d = c_p^d - c_v^d$, with c_p^v and c_p^d the specific heat at constant pressure respectively for vapor and dry air, the above equation becomes

$$l_v\frac{dM_v}{T} + M_w c^w\frac{dT}{T} + M_v(c_p^v dT - \frac{R_v T}{e}de)\frac{1}{T} + M_d(c_p^d dT - \frac{R_d T}{p}dp)\frac{1}{T} = 0$$

Now to go ahead it is useful to assume that the air we are lifting is saturated, then the vapor pressure

e is function only of temperature, that is $e=e_s=e_s(T)$. With this assumption that, as we will see below, is not restrictive, we can write $de=de_s=\frac{de_s}{dT}dT$ and substitute it into the entropy conservation equation, obtaining

$$l_v\frac{dM_v}{T}+M_w c^w\frac{dT}{T}+M_v(c_p^v dT-R_v dT-\frac{R_v T}{e}\frac{de_s}{dT}dT)\frac{1}{T}+M_d(c_p^d dT-R_d dT-\frac{R_d T}{p}dp)\frac{1}{T}=0$$

At this point it is wise to collect all the terms multiplied by the differentials, then

$$l_v\frac{dM_v}{T}+(M_w c^w+M_v c_p^v-M_d c_p^d)\frac{dT}{T}-M_v\frac{R_v}{e_s}\frac{de_s}{dT}dT-M_d\frac{R_d}{p}dp=0\ .$$

To proceed further it is necessary to introduce the total mass of water (both vapor and liquid), that is $M=M_w+M_v$ according to which $M_w=M-M_v$. With this new parameter we obtain

$$l_v\frac{dM_v}{T}+(M c^w-M_v c^w+M_v c_p^v-M_d c_p^d)\frac{dT}{T}-M_v\frac{R_v}{e_s}\frac{de_s}{dT}dT-M_d\frac{R_d}{p}dp=0\ .$$

It is now time to introduce the Clausius-Clapeyron equation

$$\frac{de_s}{dT}=\frac{l_v}{T(\frac{V_v}{M_v}-\frac{V_w}{M_w})}\approx\frac{l_v}{T}\frac{M_v}{V_v}$$

and the rate of change of latent heat of vaporization with temperature, that is

$$\frac{dl_v}{dT}=c_p^v-c^w\ .$$

The above approximation of the Clausius-Clapeyron equation is due to the fact that usually, for the common atmospheric conditions, the specific volume of water is usually much higher than the specific volume of vapor, that is $V_v/M_v\gg V_w/M_w$.

The introduction of the Clausius-Clapeyron equation and of the rate of change of latent heat with temperature is important to reduce the number of different differentials. In fact, inserting these two relationships into the above equation, we obtain

$$l_v\frac{dM_v}{T}+(M c^w+M_v\frac{dl_v}{dT}-M_d c_p^d)\frac{dT}{T}-M_v\frac{R_v}{e_s}\frac{l_v M_v}{T\,V}-M_d\frac{R_d}{p}dp=0$$

assuming that the water vapor behaves like an ideal gas, we can use the following equation of state $e_s V=M_v R_v T$ and use it into to eliminate the water vapor e_s term into the above equation, that is

$$l_v\frac{dM_v}{T}+(M c^w+M_v\frac{dl_v}{dT}-M_d c_p^d)\frac{dT}{T}-\frac{M_v R_v l_v M_v}{M_v R_v T^2}dT-M_d\frac{R_d}{p}dp=0$$ and

$$l_v\frac{dM_v}{T}+(M c^w+M_v\frac{dl_v}{dT}-M_d c_p^d)\frac{dT}{T}-\frac{l_v M_v}{T^2}dT-M_d\frac{R_d}{p}dp=0\ .$$

At this point we can reorganize all the terms into the entropy conservation equation obtaining

$$l_v\frac{dM_v}{T}+M_v\frac{dl_v}{T}-\frac{l_v M_v}{T^2}dT+(M c^w-M_d c_p^d)\frac{dT}{T}-M_d\frac{R_d}{p}dp=0\ .$$

the first three terms are the partial differentials of $d(l_v M_v/T)$, then we can write

$$d(\frac{l_v M_v}{T})+(M c^w-M_d c_p^d)\frac{dT}{T}-M_d\frac{R_d}{p}dp=0\ .$$

If now we define the total water mixing ratio (vapor plus liquid) $r_t\equiv M/M_d$ that we assume might be conserved during the process, and the vapor mixing ratio as $r_v\equiv M_v/M_d$ we will have

$$d(\frac{l_v r_v}{T})+(r_t c^w-c_p^d)\frac{dT}{T}-\frac{R_d}{p}dp=0$$

At this point using the differentiation rules and defining the constant $c_p^t=r_t c^w-c_p^d$, the above equation can be written in the more compact form

$$d(\frac{l_v r_v}{T})+c_p^t d(\ln(T))-R_d d(\ln(p))=0$$

that builds a relationship between temperature and pressure in the case of an adiabatic, reversible and saturated process. Now we have to remember that our aim is that of obtaining a potential temperature, that is a temperature independent from the pressure. To describe this process we should eliminate the explicit dependence from pressure in the first therm. This can be done defining the following potential temperature

$$\Theta=T(\frac{p_0}{p})^{\frac{R_d}{c_p^t}}$$

This potential temperature is not exactly the potential temperature for dry air, because $c_p^v\neq c_p^d$ and $c_p^t\neq r_v c_p^v+c_p^d=c_p$, but these two potential temperatures slightly differ because in the usual meteorological conditions $r_t\ll 1$ and $c_p^t\simeq r_v c_p^v+c_p^d\simeq c_p^d$.

Taking the logarithm and differentiating the above potential temperature we will have

$$d(\ln(\Theta))=d(\ln(T))-\frac{R_d}{c_p^t}d(\ln(p))$$

that can be used to substitute the pressure and temperature terms in the entropy conservation equation, that is

$$d(\frac{l_v r_v}{T})+c_p^t d(\ln(\Theta))=0$$

This equation can be easily solved integrating from the initial state characterized by r_{vi}, T and Θ to the final state characterized by $r_{vf}=0$, T_f and Θ_f. After the integration we obtain

$$c_p^t \ln(\frac{\Theta_f}{\Theta_i})+(\frac{l_v r_{vf}}{T_f}-\frac{l_v r_{vi}}{T_i})=0$$

then

$$c_p^t \ln(\frac{\Theta_f}{\Theta_i})=(\frac{l_v r_{vi}}{T_i})$$

The quantity Θ_f is the final value of the potential temperature once the air parcel had been brought up to an altitude where all its water vapor is condensed, keeping the result of condensation (liquid water) into the air parcel during the ascent. This quantity, for the above reasons, is called equivalent potential temperature and has the form

$$\Theta_e=\Theta\, e^{c_p^t(\frac{l_v r_v}{T})}$$

where T, r_v and Θ are, respectively, the current parcel temperature, vapor mixing ratio and potential temperature (but pay attention that this is just an approximation as told above). This form of the equivalent potential temperature is not the most general, because, as you probably remember, it was determined assuming that the initial parcel was saturated, i.e., its vapor pressure was that of saturation for its temperature. A more general form, useful even for unsaturated parcels, can be obtained assuming that the unsaturated parcel is lifted adiabatically up to the level at which condensation occurs (usually called Lifting Condensation Level – LCL). Then, when condensation occurs and the parcel becomes saturated, we can obtain its equivalent potential temperature using the already obtained equation for saturated processes. In other words the equivalent potential temperature of an unsaturated parcel is the equivalent potential temperature obtained starting from its lifting condensation level (LCL) which, in turn, is function of the current parcel pressure, temperature and vapor mixing ratio. In its compact form the equivalent potential temperature for an unsaturated parcel becomes

$$\Theta_e=\Theta_{LCL}(p,T,r_v)\cdot e^{c_p^t(\frac{l_v r_v}{T_{LCL}})}$$

The determination of the lifting condensation potential temperature Θ_{LCL} is not a real problem, in fact because during the adiabatic ascent the potential temperature Θ is conserved, we can say that $\Theta\equiv\Theta_{LCL}$. The only problem, approximations apart, is the determination of T_{LCL}, that is the temperature at the LCL. This can be done using the Γ_{dew} and Γ_{dry} vertical lapse rates or recursively (more precise).

Just taking the analytical approach we can write $T_{dew}(z)=\Gamma_{dew}\cdot z+T_{dew}(0)$ and $T_{dry}(z)=\Gamma_{dry}\cdot z+T_{dew}(0)$, because at the LCL temperature and dew point temperature must coincide, we are going to have $T_{dry}(z)\equiv\Gamma_{dew}(z)$ then

$$z_{lcl}=\frac{T(0)-T_{dew}(0)}{\Gamma_{dew}-\Gamma_{dry}}$$

using the so far obtained z_{lcl} in one of the above equations that describe the vertical temperatures trend, we will have

$$T_{lcl}=\Gamma_{dry}\cdot\frac{T(0)-T_{dew}(0)}{\Gamma_{dew}-\Gamma_{dry}}+T(0)$$

which gives us the temperature at the lifting condensation level once are known the dew point and thermal vertical lapse rates as well as the current temperature of the parcel. This T_{lcl} can be used in the equivalent potential temperature equation

$$\Theta_e=\Theta\cdot e^{{c^t}_p(\frac{l_v r_v}{T_{LCL}})}\ .$$