Per quali $\alpha \in \mathbb{R}$ e $N \in \mathbb{N}^+$, esiste finito

$$\iint_{J_\alpha} (x^2+y^2)^{-N}\,dx\,dy$$

con $J_\alpha = \{(x,y) : 0<x, \quad 0<y<x^\alpha\}$.

Osservazione

Per ogni $N \in \mathbb{N}^+$,

$$\iint_B (x^2+y^2)^{-N}\,dx\,dy = +\infty$$

con $B = \{(x,y) : 0<x^2+y^2<1\}$.

Quindi, per confronto, per ogni $N \in \mathbb{N}^+$,

$$\iint_Q (x^2+y^2)^{-N}\,dx\,dy = +\infty, \qquad \text{con } Q =]0,1[\times]0,1[$$

e $$\iint_T (x^2+y^2)^{-N}\,dx\,dy = +\infty, \qquad \text{con } T = \{(x,y) : 0<x<1,\ 0<y<1\}.$$

Discussione

- $\alpha<0$: $\quad Q \subseteq J_\alpha \implies \iint_{J_\alpha} (x^2+y^2)^{-N}\,dx\,dy = +\infty, \ \forall N \in \mathbb{N}^+$
- $\alpha=0$: $\quad Q \subseteq J_0 \implies \iint_{J} (x^2+y^2)^{-N}\,dx\,dy = +\infty, \ \forall N \in \mathbb{N}^+$
- $0<\alpha<1$: $\quad T \subseteq J_\alpha \implies \iint_{J_\alpha} (x^2+y^2)^{-N}\,dx\,dy = +\infty, \ \forall N \in \mathbb{N}^+$
- $\alpha=1$: $\quad T \subseteq J_1 \implies \iint_{J_1} (x^2+y^2)^{-N}\,dx\,dy = +\infty, \ \forall N \in \mathbb{N}^+$
- $\alpha>1$: $\quad J_\alpha = H_\alpha \cup K_\alpha,$

con $\quad H_\alpha = \{(x,y) : 0<x<1, \quad 0<y<x^\alpha\}$

$\qquad K_\alpha = \{(x,y) : x \geq 1, \quad 0<y<x^\alpha\}$

• $$\iint_{[0,+\infty[^2\setminus B} (x^2+y^2)^{-\nu}\,dx\,dy \le \iint_{K_\alpha} (x^2+y^2)^{-\nu}\,dx\,dy \le \iint_{\mathbb{R}^2\setminus B} (x^2+y^2)^{-\nu}\,dx\,dy,$$

usando coordinate polari, si vede che il I e il III integrale esistono finiti se e solo se $\nu > 1$.

• $$\iint_{H_\alpha} (x^2+y^2)^{-\nu}\,dx\,dy = \int_0^1 \left(\int_0^{x^\alpha} (x^2+y^2)^{-\nu}\,dy \right) dx$$

$$\le \int_0^1 x^{\alpha-2\nu}\,dx < +\infty \iff \alpha - 2\nu > -1$$

$$\iff \alpha > 2\nu - 1 \ (>1)$$

D'altra parte, $$\iint_{H_\alpha} (x^2+y^2)^{-\nu}\,dx\,dy \ge \int_0^1 \left(\int_0^{x^\alpha} (x^2+x^{2\alpha})^{-\nu}\,dy \right) dx$$

$$= \int_0^1 (x^2+x^{2\alpha})^{-\nu} x^\alpha\,dx \underset{\substack{\uparrow \\ \alpha>1}}{\ge} 2^{-\nu} \int_0^1 x^{\alpha-2\nu}\,dx = +\infty$$

$$\iff \alpha - 2\nu \le -1$$

$$\iff \alpha \le 2\nu - 1$$

In conclusione,

$$\iint_{J_\alpha} (x^2+y^2)^{-\nu}\,dx\,dy \quad \text{esiste finito}$$

se e solo se $\quad \nu > 1 \quad$ e $\quad \alpha > 2\nu - 1$

Calcolare $g(t)=\int_0^{+\infty} e^{-x}\frac{\sin(tx)}{x}\,dx$, con $t\in\mathbb{R}$.

Posto $I=[-a,a]$, $J=]0,+\infty[$ e $f(t,x)=e^{-x}\frac{\sin(tx)}{x}$,

si ha $\frac{\partial f}{\partial t}(t,x)=e^{-x}\cos(tx)$, con

$$|f(t,x)|\le e^{-x}t\le a e^{-x}=\varphi(x)$$

e

$$\left|\frac{\partial f}{\partial t}(t,x)\right|\le e^{-x}=\psi(x) \quad \text{in } I\times J.$$

Quindi g è di classe C^1 in I e, per l'arbitrarietà di a, su $\mathbb{R}$. Inoltre risulta

$$g'(t)=\int_0^{+\infty} e^{-x}\cos(tx)\,dx=\left[-e^{-x}\cos(tx)\right]_0^{+\infty}-\int_0^{+\infty} te^{-x}\sin(tx)\,dx$$

$$=1+\left[te^{-x}\sin(tx)\right]_0^{+\infty}-\int_0^{+\infty}t^2e^{-x}\cos(tx)\,dx$$

$$=1-t^2g'(t),$$

cioè

$$g'(t)=\frac{1}{1+t^2}.$$

Poiché $g(0)=0$, si conclude che

$$g(t)=\operatorname{arctg} t.$$