Corso di Laurea in Matematica, Corso di Laurea in Fisica
Esame di Analisi 3, modulo B
A.a. 2016-2017, sessione estiva, II appello

COGNOME ______ **NOME** ______

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Corso di Studi: Matematica ○ Fisica ○

ESERCIZIO N. 1. Si ponga $B = \{(x,y,z,t) \in \mathbb{R}^4 : x^2+y^2+z^2+t^2 \leq 1\}$ e, per ogni $t \in]-1,1[$, $B_t = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 \leq 1-t^2\}$.

(i) Per ogni $t \in]-1,1[$, si calcoli $\iiint_{B_t} |zt| \, dxdydz$.

$$\iiint_{B_t} |zt|\,dxdydz = |t| \int_0^{\pi}\left(\int_0^{2\pi}\left(\int_0^{\sqrt{1-t^2}} |\rho\cos\varphi|\rho^2 \sin\varphi \, d\rho\right)d\vartheta\right)d\varphi$$

$$= |t| 2\pi \left(\int_0^{\pi} |\cos\varphi| \sin\varphi \, d\varphi\right)\cdot\left(\int_0^{\sqrt{1-t^2}} \rho^3 \, d\rho\right)$$

$$= |t| 2\pi \left[\sin^2\varphi\right]_0^{\frac{\pi}{2}} \cdot \left[\frac{\rho^4}{4}\right]_0^{\sqrt{1-t^2}}$$

$$= \frac{\pi}{2}|t|(1-t^2)^2$$

(ii) Si calcoli $\iiiint_B |zt| \, dxdydzdt$.

$$\iiiint_B |zt|\,dxdydzdt = \int_{-1}^{1}\left(\iiint_{B_t} |zt|\,dxdydz\right)dt$$

$$= \int_{-1}^{1} \frac{\pi}{2}|t|(1-t^2)^2\,dt$$

$$= \frac{\pi}{2}\int_0^1 2t(1-t^2)^2\,dt$$

$$= \frac{\pi}{2}\left[\frac{1}{3}(1-t^2)^3\right]_0^1 = \frac{\pi}{6}$$

ESERCIZIO N. 2. Si ponga, per ogni $\alpha > 0$,

$$J_\alpha = \{(x, y, z) \in \mathbb{R}^3 : 0 \le z \le (x^2 + y^2)^{-\alpha} - 1\}.$$

(*i*) Si stabilisca per quali $\alpha > 0$, J_α ha volume finito.

- $(x^2+y^2)^{-\alpha} - 1 \geq 0 \iff x^2+y^2 \leq 1$, essendo $\alpha > 0$.
- posto per ogni n, $A_n = \{(x,y,z) : \frac{1}{n^2} \leq x^2+y^2 \leq 1, 0 \leq z \leq (x^2+y^2)^{-\alpha} - 1\}$ si ha

$$\iiint_{A_n} 1 \, dx\, dy\, dz = \int_0^{2\pi} \left(\int_{\frac{1}{n}}^1 \left(\int_0^{(x^2+y^2)^{-\alpha}-1} 1 \, dz \right) \rho \, d\rho \right) d\vartheta$$

$$= 2\pi \int_{\frac{1}{n}}^1 (\rho^{1-2\alpha} - \rho) \, d\rho$$

- $\lim_{n\to+\infty} \int_{\frac{1}{n}}^1 (\rho^{1-2\alpha} - \rho) \, d\rho$ è finito $\iff 1 - 2\alpha > -1$

$\iff \alpha < 1$.

(*ii*) Si stabilisca per quali $\alpha > 0$, la frontiera di J_α ha area finita.

- Si ha, per ogni n,

$$\iint_{A_n} \sqrt{1 + \left(2\alpha (x^2+y^2)^{-\alpha-1} x\right)^2 + \left(2\alpha (x^2+y^2)^{-\alpha-1} y\right)^2} \, dx\, dy$$

$$= \iint_{A_n} \sqrt{1 + 4\alpha^2 (x^2+y^2)^{-2\alpha-1}} \, dx\, dy$$

$$= \int_0^{2\pi} \left(\int_{\frac{1}{n}}^1 \sqrt{1 + 4\alpha^2 \rho^{-4\alpha-2}} \, \rho \, d\rho \right) d\vartheta = 2\pi \int_{\frac{1}{n}}^1 \rho^{-2\alpha} \sqrt{\rho^{4\alpha+2} + 4\alpha^2} \, d\rho$$

- $\lim_{n\to+\infty} \int_{\frac{1}{n}}^1 \rho^{-2\alpha} \sqrt{\rho^{4\alpha+2} + 4\alpha^2} \, d\rho$ è finito $\iff -2\alpha > -1$

$\iff \alpha < \frac{1}{2}$