Corso di Laurea in Matematica, Corso di Laurea in Fisica
Esame di Analisi 3, modulo B
A.a. 2017-2018, I prova intermedia

COGNOME ______________________ NOME ______________________

N. Matricola ______________________ Anno di corso ______________________

Corso di Studi: Matematica ○ Fisica ○

ESERCIZIO N. 1. Si ponga $D = \{(x,y) \in \mathbb{R}^2 : 0 \le y \le 1 + 2x - x^2,\ 0 \le x \le 1\}$ e si indichino con E_x ed E_y i solidi ottenuti facendo facendo ruotare D di 2π intorno all'asse x e all'asse y, rispettivamente.

(*i*) Si calcoli $m_2(D)$.

$$m_2(D) = \iint_D 1\,dx\,dy = \int_0^1 \left(\int_0^{1+2x-x^2} 1 \cdot dy \right) dx$$
$$= \int_0^1 (1+2x-x^2)\,dx$$
$$= \left[x + x^2 - \tfrac{1}{3}x^3 \right]_0^1 = \frac{5}{3}$$

(*ii*) Si calcoli $m_3(E_x)$.

$$m_3(E_x) = 2\pi \iint_D y\,dx\,dy = 2\pi \int_0^1 \left(\int_0^{1+2x-x^2} y\,dy \right) dx$$
$$= \pi \int_0^1 (1+2x-x^2)^2\,dx$$
$$= \pi \left[x + 2x^2 + \tfrac{2}{3}x^3 - x^4 + \tfrac{1}{5}x^5 \right]_0^1 = \frac{43}{15}\pi$$

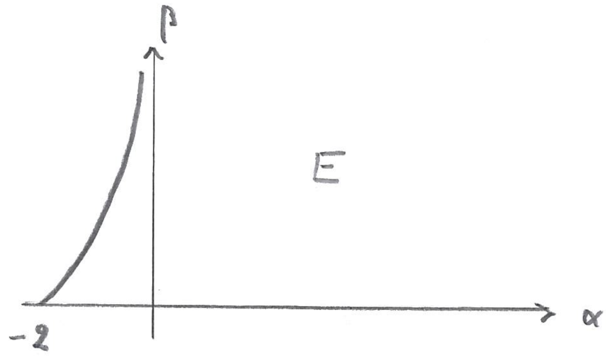
(*iii*) Si calcoli $m_3(E_y)$.

$$m_3(E_y) = 2\pi \iint_D x\,dx\,dy = 2\pi \int_0^1 \left(\int_0^{1+2x-x^2} x\,dy \right) dx$$
$$= 2\pi \int_0^1 x(1+2x-x^2)\,dx$$
$$= 2\pi \left[\tfrac{1}{2}x^2 + \tfrac{2}{3}x^3 - \tfrac{1}{4}x^4 \right]_0^1 = \frac{11}{6}\pi$$

ESERCIZIO N. 2. Si definiscano, per $\alpha \in \mathbb{R}$, $J_\alpha = \{(x,y) \in \mathbb{R}^2 : 0 < x < 1, 0 < y < x^\alpha\}$ e, per $\beta \in [0,+\infty[$, $f_\beta(x,y) = xy^\beta$. Si determinino e si rappresentino nel piano le coppie (α,β) per cui f_β è integrabile in senso generalizzato su J_α.

RISULTATO

$$E = \left\{(\alpha,\beta) : \alpha > \frac{-2}{\beta+1}, \beta \geq 0\right\}$$



SVOLGIMENTO

Posto $A_n = \{(x,y) : \frac{1}{n} < x < 1, 0 < y < x^\alpha\}$, si ha

$$\iint_{A_n} xy^\beta \,dx\,dy = \int_{\frac{1}{n}}^{1} \left(\int_0^{x^\alpha} xy^\beta \,dy\right) dx$$

$$= \int_{\frac{1}{n}}^{1} \frac{1}{\beta+1} x \left[y^{\beta+1}\right]_0^{x^\alpha} dx$$

$$= \frac{1}{\beta+1} \int_{\frac{1}{n}}^{1} x^{\alpha\beta+\alpha+1} dx .$$

Esiste finito $\lim_{n\to+\infty} \iint_{A_n} xy^\beta \,dx\,dy$ se e solo se

$$\alpha\beta + \alpha + 1 > -1 \iff \alpha > -\frac{2}{\beta+1}.$$

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ESERCIZIO N. 3. Posto, per $n \in \mathbb{N}^+$,

$$A_n = \left\{(x,y,z) \in \mathbb{R}^3 : \frac{1}{n} \le x+y+z \le 2, \frac{n}{n+1} \le x-y+z \le \frac{3n}{n+1}, \frac{1}{n} \le x+y-z \le 2n\right\},$$

si calcoli

$$\lim_{n\to+\infty} \iiint_{A_n} \frac{e^z}{e^{x+y}\sqrt{x+y+z}}\,dx\,dy\,dz.$$

RISULTATO

$\sqrt{2}$

SVOLGIMENTO

Posto

$$\begin{cases} u = x+y+z \\ v = x-y+z \\ w = x+y-z \end{cases} \quad \text{e} \quad M = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \quad \text{si ha}$$

$$M\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} u \\ v \\ w \end{pmatrix} \quad \text{e} \quad M^{-1}\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

Posto

$$B_n = \left\{(u,v,w) : \frac{1}{n} \le u \le 2, \frac{n}{n+1} \le v \le \frac{3n}{n+1}, \frac{1}{n} \le w \le 2n\right\}$$

e osservato che $\det M^{-1} = (\det M)^{-1} = \frac{1}{4}$, risulta

$$\iiint_{B_n} \frac{1}{4}\frac{e^{-w}}{\sqrt{u}}\,du\,dv\,dw = \frac{1}{4}\int_{\frac{1}{n}}^{2} \frac{1}{\sqrt{u}}\,du \cdot \int_{\frac{n}{n+1}}^{\frac{3n}{n+1}} 1\,dv \cdot \int_{\frac{1}{n}}^{2n} e^{-w}\,dw$$

$$= \frac{1}{4}\left(2\sqrt{2} - 2\sqrt{\frac{1}{n}}\right) \cdot \frac{2n}{n+1} \cdot \left(-e^{-2n} + e^{-\frac{1}{n}}\right) \to \sqrt{2}$$

se $n \to +\infty$

e quindi $\displaystyle\lim_{n\to+\infty} \iiint_{A_n} \frac{e^{z-x-y}}{\sqrt{x+y+z}}\,dx\,dy\,dz = \sqrt{2}.$

ESERCIZIO 4. (facoltativo) Si definisca $f : R = [0,1]^2 \to \mathbb{R}$ ponendo

$$f(x,y) = \begin{cases} \dfrac{1}{y^2} & \text{se } 0 < x < y < 1, \\ -\dfrac{1}{x^2} & \text{se } 0 < y < x < 1, \\ 0 & \text{altrimenti.} \end{cases}$$

Si stabilisca:

(i) se f è localmente integrabile in R

Posto $A_n = \{(x,y) : \frac{1}{n} \le x \le 1,\ \frac{1}{n} \le y \le 1\}$. $(A_n)_n$ è una successione invadente R adatta a f.

(ii) se esiste $\displaystyle\iint_R f(x,y)\,dx\,dy$

$$\iint_{A_n} |f(x,y)|\,dx\,dy = \int_{\frac{1}{n}}^{1} \left(\int_{\frac{1}{n}}^{1} |f(x,y)|\,dy \right) dx = \int_{\frac{1}{n}}^{1} \left(\int_{\frac{1}{n}}^{x} \frac{1}{x^2}\,dy + \int_{x}^{1} \frac{1}{y^2}\,dy \right) dx$$

$$= \int_{\frac{1}{n}}^{1} \left(\frac{1}{x^2}\left(x - \frac{1}{n}\right) - 1 + \frac{1}{x} \right) dx = \int_{\frac{1}{n}}^{1} \left(\frac{2}{x} - \frac{1}{nx^2} - 1 \right) dx = \left[2\log x + \frac{1}{nx} - x \right]_{\frac{1}{n}}^{1} \to +\infty,$$

se $n \to +\infty$

(iii) se esiste $\displaystyle\int_0^1 \Big(\int_0^1 f(x,y)\,dy \Big) dx$

$$\int_0^1 f(x,y)\,dy = \int_0^x -\frac{1}{x^2}\,dy + \int_x^1 \frac{1}{y^2}\,dy = -\frac{1}{x} + \left(\frac{1}{x} - 1 \right) = -1$$

$$\int_0^1 \left(\int_0^1 f(x,y)\,dy \right) dx = -1$$

(iv) se esiste $\displaystyle\int_0^1 \Big(\int_0^1 f(x,y)\,dx \Big) dy$

$$\int_0^1 f(x,y)\,dx = \int_0^y \frac{1}{y^2}\,dx + \int_y^1 -\frac{1}{x^2}\,dx = \frac{1}{y} + \left(1 - \frac{1}{y} \right) = 1$$

$$\int_0^1 \left(\int_0^1 f(x,y)\,dx \right) dy = 1$$

(v) quali dei precedenti integrali sono uguali fra loro.

Nessuno