The dynamics of surface waves（linear theory）

We derive some basic relations to surface gravity waves, which are most fundamental movement of water. Though real waves have strong non-linearity, we consider only linear waves. Also surface tensions are not considered.

Consider a vertical cross section (*x, z*), where *x* is the direction wave propagates, and *z* is set upward, with the zero base of mean water level. The movement is assumed to be uniform in the *y* direction. At first approximation, fluid is assumed to be **inviscid**.

Waves generated from static state by gravity effect are **irrotational**, so we can define velocity potential φ.

$$\mathbf{u} = \nabla \phi. \tag{1}$$

In the **incompressible** fluid such as water, continuity equations is

$$\nabla \cdot \mathbf{u} = 0. \tag{2}$$

Therefore it follows **Laplace's equation**.

$$\nabla^2 \phi = 0. \tag{3}$$

Next, let's consider boundary conditions. At the bottom floor, water cannot pass into underground, the normal velocity component vanishes.

$$(w)_{z=-d} = \left(\frac{\partial \phi}{\partial z}\right)_{z=-d} = 0. \tag{4}$$

At the free surface of water $z = \zeta(x, z)$, water parcels are assumed as not to pass through its surface and keep in surface without wave breaking,

$$w = \frac{D\eta}{Dz} = \frac{\partial \eta}{\partial t} + u \cdot \frac{\partial \eta}{\partial x}. \tag{5}$$

Using velocity potential, a kinematic boundary condition at the free surface is

$$\left(\frac{\partial \phi}{\partial z}\right)_{z=\eta} = \frac{\partial \eta}{\partial t} + \left(\frac{\partial \phi}{\partial x}\right)_{z=\eta} \cdot \frac{\partial \eta}{\partial x}. \tag{6}$$

On the other hand, a dynamical boundary condition at the free surface is derived like this. Using the Bernoulli equation, the pressure at any point is expressed as,

$$\frac{P}{\rho}=\frac{P_0}{\rho}-\frac{\partial\phi}{\partial t}-\frac{1}{2}\left\{\left(\frac{\partial\phi}{\partial x}\right)^2+\left(\frac{\partial\phi}{\partial z}\right)^2\right\}-g\eta+F(t), \tag{7}$$

where ρ water density. Supposing that atmospheric pressure P_0 is constant, combining additional function $F(t)$ with φ, the dynamical boundary condition at the free surface is,

$$\left(\frac{\partial\phi}{\partial t}\right)_{z=\eta}+\frac{P}{\rho}+g\eta+\frac{1}{2}\left\{\left(\frac{\partial\phi}{\partial x}\right)^2+\left(\frac{\partial\phi}{\partial z}\right)^2\right\}_{z=\eta}=0, \tag{8}$$

These four equations, Laplace's equation (3) and boundary conditions (4), (6), (8), are the governing equations.

It is difficult to solve these equations rigorously, because non-linear terms are involved in the equations. For simplicity, wave amplitude is supposed to be smaller than wave length (this assumption is generally good approximation in real waves in oceans), equations are expressed by Taylor series expansions about $z = 0$.

For example,

$$\left(\frac{\partial\phi}{\partial z}\right)_{z=\eta}=\left(\frac{\partial\phi}{\partial z}\right)_{z=0}+\left\{\frac{\partial}{\partial z}\left(\frac{\partial\phi}{\partial z}\right)\right\}_{z=0}\eta+\frac{1}{2}\left\{\frac{\partial^2}{\partial z^2}\left(\frac{\partial\phi}{\partial z}\right)\right\}_{z=0}\eta^2+\cdots. \tag{9}$$

If we keep just first order term, we can get the equation set for linear waves,

Laplace's equation

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial z^2}=0. \tag{10}$$

boundary condition at the bottom

$$\left(\frac{\partial\phi}{\partial z}\right)_{z=-d}=0. \tag{11}$$

kinematic boundary condition at the free surface

$$\left(\frac{\partial\phi}{\partial z}\right)_{z=0}=\frac{\partial\eta}{\partial t}. \tag{12}$$

dynamical boundary condition at the free surface

$$\left(\frac{\partial\phi}{\partial t}\right)_{z=0}+g\eta=0 \tag{13}$$

Using equations (12) and (13), η is deleted as,

$$\left(\frac{\partial^2\phi}{\partial t^2}\right)_{z=0} = -g\left(\frac{\partial\phi}{\partial z}\right)_{z=0} \tag{14}$$

Now consider the wave like solution toward the x direction. Velocity potential is supposed to be expressed with exponential form as

$$\eta = Z(z)\cdot e^{i(kx-\omega t)} \tag{15}$$

Putting it into equation (10) gives

$$\frac{d^2Z}{dt^2} - k^2 Z = 0 \tag{16}$$

This differential equation has a general solution and expressed with constant A and B,

$$Z = Ae^{kz} + Be^{-kz} \tag{17}$$

Putting equations (15), (17) into (11), (14) leads to equations about A and B as,

$$\begin{cases} Ae^{-kd} - Be^{kd} = 0 \\ (\omega^2 - gk)A + (\omega^2 + gk)B = 0 \end{cases} \tag{18}$$

If A and B have a non-trivial solution, It need to be

$$\begin{vmatrix} e^{-kd} & -e^{kd} \\ \omega^2 - gk & \omega^2 + gk \end{vmatrix} = 0 \tag{19}$$

From this determinant,

$$\omega^2 = gk\frac{e^{kd} - e^{-kd}}{e^{kd} + e^{-kd}} = gk\tanh kd \tag{20}$$

This is the **dispersion relation**, and quite an important equation. With this relation and basic wave relations, some fundamental values are defined.

phase speed:

$$c = \frac{\omega}{k} = \sqrt{\frac{g}{k}\tanh kd} = \sqrt{\frac{gL}{2\pi}\tanh\frac{2\pi d}{L}}, \tag{21}$$

wave length:

$$L = cT = \sqrt{\frac{gT^2}{k}\tanh kd} = \frac{gT^2}{2\pi}\tanh\frac{2\pi d}{L}, \tag{22}$$

group velocity:

$$c_g = \frac{d\omega}{dk} = \frac{c}{2}\left(1 + \frac{2kd}{\sinh 2kd}\right), \tag{23}$$

★shallow water waves (long waves) and deep water waves (short waves)

The derived relations are rather complicated expressions with hyperbolic functions. By considering the limit toward shallow and deep waters, the forms become much simpler.

①shallow water waves

In the limit of water depth $d \to 0$, $\tanh kd / kd = 1$ etc, the relation becomes

$$\begin{cases} c = \sqrt{gd\dfrac{\tanh kd}{kd}} = \sqrt{gd} \\ L = \sqrt{gdT^2} = T\sqrt{gd} \\ c_g = \dfrac{c}{2}\left(1 + \dfrac{2kd}{\sinh 2kd}\right) = \dfrac{c}{2}(1+1) = c \end{cases}, \tag{24}$$

The phase speed is only a function of water depth, not wave length. The group velocity coincides with its phase velocity, and not a function of wave length (wave number), waves are not dispersive.

The reliability of this relation simply depends on accuracy of approximation of hyperbolic functions. In general, if the water depth is smaller than one twenty-fifth ($d < L/25$), then the value is almost rigorous one. The real wave length of storm surges and tsunamis is about several tens km, on the contrary water depth at coasts are at most 500 m, it is reasonable to consider these waves as shallow water waves (long waves).

②deep water waves

In the limit to $d \to \infty$, $\tanh kd \to 1$, $\sinh kd / kd \to \infty$ etc,

$$
\begin{cases}
c = \sqrt{\dfrac{g}{k}} = \sqrt{\dfrac{gL}{2\pi}} = \dfrac{gT}{2\pi} \\
L = \dfrac{gT^2}{2\pi} \\
c_g = \dfrac{c}{2}\left(1 + \dfrac{2kd}{\sinh 2kd}\right) = \dfrac{c}{2}
\end{cases}
, \qquad (25)
$$

Different from long waves, the phase speed is only a function of wavelength (or wave periods, frequency), not water depth. The group velocity is half of its phase speed. Such waves are dispersive.

This relation is almost rigorous where water depth is deeper than half a wavelength. Wavelengths of ocean waves are about 100m or so, then almost in whole ocean, just without coastal area where water depth becomes shallow, ocean wave are considered as short waves.

These relations are summarized in Table A.1.

Table A.1 relations of external (linear) gravity waves

	Deep water waves ($h > L/2$)	Theoretical expressions	Shallow water waves ($h < L/25$)
Phase speed C	$\sqrt{\dfrac{gL}{2\pi}}$	$\sqrt{\dfrac{gL}{2\pi}\tanh\left(\dfrac{2\pi h}{L}\right)}$	$\sqrt{gh}$
Wavelength L	$\dfrac{gT^2}{2\pi}$	$\dfrac{gT^2}{2\pi}\tanh\left(\dfrac{2\pi h}{L}\right)$	$T\sqrt{gh}$
Group velocity C_g	$\dfrac{1}{2}\sqrt{\dfrac{gL}{2\pi}} = \dfrac{C}{2}$	$\dfrac{1}{2}C\left\{1 + \dfrac{4\pi h/L}{\sinh(4\pi h/L)}\right\}$	$\sqrt{gh} = C$

All relations are expressed with

g : gravitational acceleration L : wave length h : water depth T : wave period