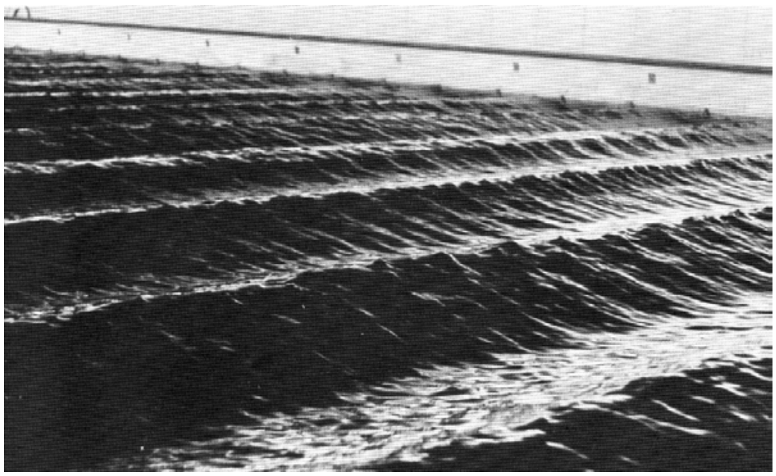
Higher-order Stokes theory

Linear (Airy) theory is adequate to describe the primary motions produced by deep-water waves. However, most real waves are not sinusoidal, but complex periodic features (see photo below from Van Dyke's *Album of Fluid Motion*).



Therefore, we will add the next term in the general expansion described in Equations (4,7-9) of the last lecture.

Doing so, we find –

$$\eta = \frac{H}{2}\cos 2\pi(kx-\omega t)+\frac{\pi H^2}{2L}\frac{\cosh(kh)[2+\cosh(2kh)]}{[\sinh(kh)]^3}\cos[2(kx-\omega t)]$$

(1)

where the celerity C is defined by

$$C = \frac{gT}{2\pi}\tanh\left(\frac{2\pi h}{L}\right)\left[1+\left(\frac{\pi H}{L}\right)^2 \frac{5+2\cosh(4\pi h/L)+2\cosh^2(4\pi h/L)}{8\sinh^4(2\pi h/L)}\right]$$

(2)

Examining the deep-water case (i.e., where $H/h << 1$) allows us to see what the equations are doing.

$$\eta_\infty = \frac{H_\infty}{2}\cos 2\pi\left(\frac{x}{L_\infty}-\frac{t}{T}\right)+\frac{\pi H_\infty^2}{4L_\infty}\cos 4\pi\left(\frac{x}{L_\infty}-\frac{t}{T}\right) \qquad (3)$$

where L_∞ is defined in the first lecture –

$$L_\infty = \frac{gT^2}{2\pi} \qquad (4)$$

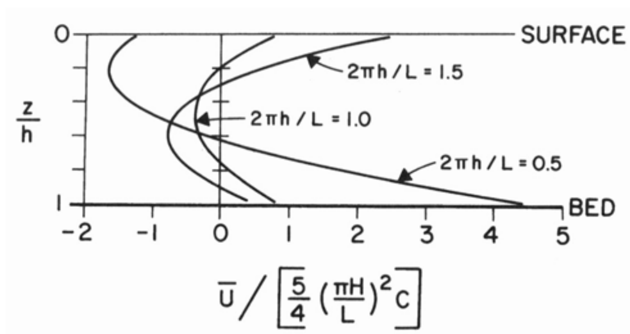
Stokes drift

One of the most important effects from the addition of second-order terms is the incomplete closure of particle paths (unlike the photograph from the first lecture). That is, there is a net velocity in the direction of wave propagation. This velocity is called **Stokes drift** and is formulated from the second-order equations to be –

$$\overline{U} = \frac{1}{2}\left(\frac{\pi H}{L}\right)^2 \left[C \frac{\cosh[2k(z+h)]}{\sinh^2 kh} \right] \tag{5}$$

Remember, this result is for inviscid, irrotational flow only. Most natural situations where this will be important, at least one of these effects will be important.

Longuet-Higgins (1953) investigated the viscous case in a finite domain. Their results can be summarized in the figure –



Where the function represented in the figure has the form

$$\frac{\overline{U}}{1.25(\pi H/L)^2 C} = \frac{1}{4\sinh^2 kh}\Big[2\cosh 2kh(\mu - 1) + 3$$

$$kh \sinh 2kh\left(3\mu^2 - 4\mu + 1\right) + 3\left(\frac{\sinh 2kh}{2kh} + \frac{3}{2}\right)\left(\mu^2 - 1\right)\Big] \tag{6}$$

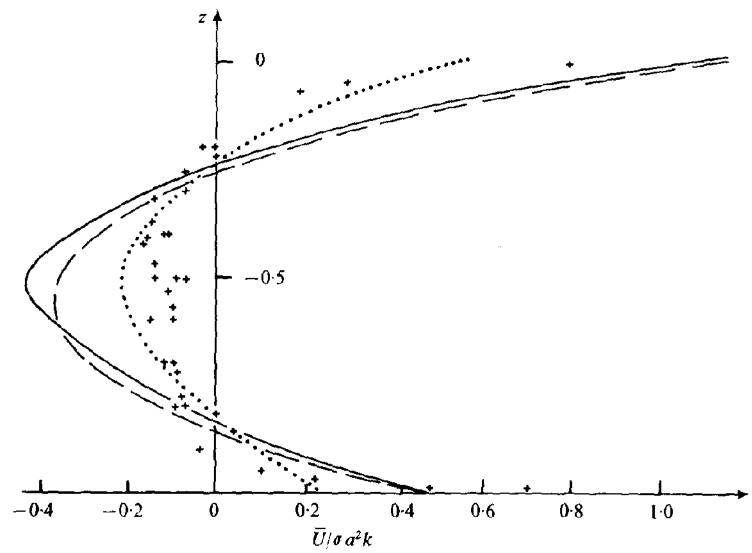
where $\mu = z/h$.

Near the bed, the shoreward velocity can be expressed more simply

$$\overline{U_0} = \frac{1.25(\pi H/L)^2 C}{\sinh^2 kh} \tag{7}$$

Longuet-Higgins simplified transport of momentum in the interior of the flow dramatically to arrive at (6) and the plots in the figure. Both theory and experiment have struggled to capture the quantitative aspects of true "Stokes drift".

Most experiments and theory capture in a qualitative sense of the vertical distribution of the Stokes drift velocity (see below – Fig. 5, Liu and Davies, 1977). Other theories and experimental data sets have been put forth (Craik, 1982; Liu and Davies, 1977; Mei et al., 1972). None of them can match the laboratory data that is considered most realistic (Russell and Osorio, 1957). As Craik (1982) states, "a truly definitive experiment on drift profiles is still lacking, so long after Stokes's pioneering paper." In the future, Lagrangian attempts may hold some promise (Phillips, 2001).



However, for our purposes, all of these theories reflect the same diffusion from the boundaries seen in the original work of Longuet-Higgins (1953).

Solitary waves and the KdV equation

Solitary waves were first noted by Russell (1834). As a result, he set out to perform simple experiments to obtain their celerity. His result –

$$C = \sqrt{g(h+H)} \qquad (8)$$

was discovered in 1844 – and is still used today for waves that are essentially non-periodic.

Following Russell's work, both Boussinesq (1871) and Rayleigh (1876) derived Russell's work from first principles. In doing so, they also found

$$\eta = H \operatorname{sech}^2[\beta(x - Ct)] \quad (9)$$

where C is defined above and

$$\beta^2 = \frac{3H}{4h^2(h+H)} \quad (9a)$$

Unfortunately, this result is only true in the limit of $H/h \to 0$.

A few studies have postulated a slight correction, which accounts for the slight variation in the celerity with dimensionless depth. That correction is

$$C = \sqrt{gh}\left[1 + \frac{1}{2}\left(\frac{H}{h}\right) - \frac{3}{30}\left(\frac{H}{h}\right)^2\right] \quad (10)$$

KdV theory

Korteweg and de Vries (1895), in a truly classic work, developed the equation from which (6) and (7) result. Their equation permits periodic solutions and is relevant to innumerable physical applications.

In fact, it is simplest equation that encompasses both nonlinearity and dispersion. Korteweg and de Vries called the waves described by their equation **cnoidal**. They are also called **long** waves.

Their equation has the general form

$$\frac{\partial\eta}{\partial t}+\eta\frac{\partial\eta}{\partial x}+\frac{\partial^3\eta}{\partial x^3}=0 \qquad (11)$$

A good reference describing KdV mathematical development and its numerous potential applications is *Solitons* by Drazin and Johnson.

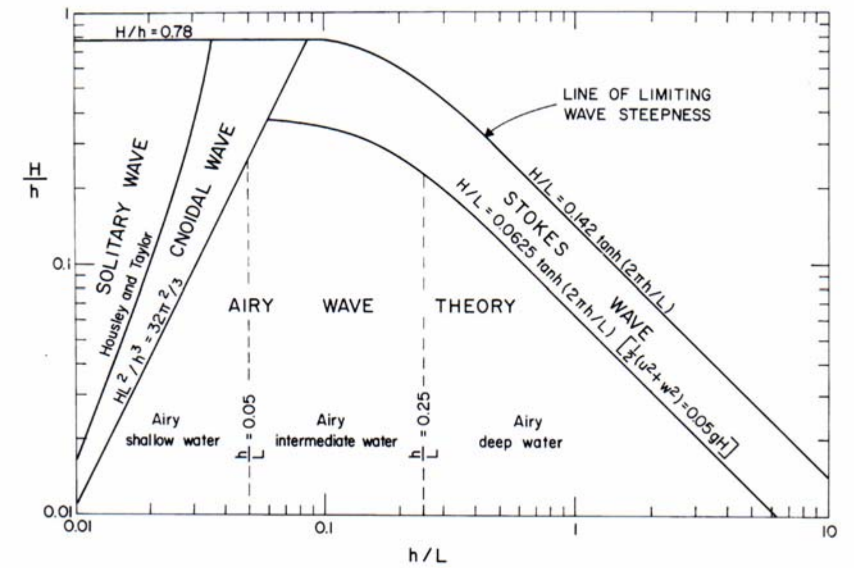
For our purposes, we are interested only in the solutions that (11) permit – most specifically, the dispersion relations. It turns out that the dispersion relations have to have the form

$$\omega = kC(k^2) \qquad (12)$$

where $C(k^2)$ is the celerity, which must be a function of k^2.

Summary

The parameter regions that are relevant to each type of wave are summarized nicely in Figure 5-29 of Komar (shown below).



H/h = 0.78
LINE OF LIMITING WAVE STEEPNESS
SOLITARY WAVE
Housley and Taylor
CNOIDAL WAVE
HL^2 / h^3 = 32π^2/3
STOKES WAVE
H/L = 0.142 tanh (2πh/L)
H/L = 0.0625 tanh (2πh/L)
[½(u² + w²) = 0.05gH]
AIRY WAVE THEORY
Airy shallow water
h/L = 0.05
Airy intermediate water
h/L = 0.25
Airy deep water
H/h
h/L
0.01
0.1
1
10