APPENDIX F

An Integral Laminar Boundary Layer Solution for Parallel Flow over a Flat Plate

An alternative approach to solving the boundary layer equations involves the use of an approximate *integral* method. The approach was originally proposed by von Kárman [1] in 1921 and first applied by Pohlhausen [2]. It is without the mathematical complications inherent in the *exact* (*similarity*) method of Section 7.2.1; yet it can be used to obtain reasonably accurate results for the key boundary layer parameters (δ, δ_t, δ_c, C_f, h, and h_m). Although the method has been used with some success for a variety of flow conditions, we restrict our attention to parallel flow over a flat plate, subject to the same restrictions enumerated in Section 7.2.1, that is, *incompressible laminar flow* with *constant fluid properties* and *negligible viscous dissipation.*

To use the method, the boundary layer equations, Equations 7.4 through 7.7, must be cast in integral form. These forms are obtained by integrating the equations in the y direction across the boundary layer. For example, integrating Equation 7.4, we obtain

$$\int_0^\delta \frac{\partial u}{\partial x}\,dy + \int_0^\delta \frac{\partial v}{\partial y}\,dy = 0 \tag{F.1}$$

or, since $v = 0$ at $y = 0$,

$$v(y = \delta) = -\int_0^\delta \frac{\partial u}{\partial x}\,dy \tag{F.2}$$

Similarly, from Equation 7.5, we obtain

$$\int_0^\delta u\frac{\partial u}{\partial x}\,dy + \int_0^\delta v\frac{\partial u}{\partial y}\,dy = \nu\int_0^\delta \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right)dy$$

or, integrating the second term on the left-hand side by parts,

$$\int_0^\delta u\frac{\partial u}{\partial x}\,dy + uv\,\Big|_0^\delta - \int_0^\delta u\frac{\partial v}{\partial y}\,dy = \nu\frac{\partial u}{\partial y}\Big|_0^\delta$$

Substituting from Equations 7.4 and F.2, we obtain

$$\int_0^\delta u\frac{\partial u}{\partial x}\,dy - u_\infty\int_0^\delta \frac{\partial u}{\partial x}\,dy + \int_0^\delta u\frac{\partial u}{\partial x}\,dy = -\nu\frac{\partial u}{\partial y}\Big|_{y=0}$$

or

$$u_\infty\int_0^\delta \frac{\partial u}{\partial x}\,dy - \int_0^\delta 2u\frac{\partial u}{\partial x}\,dy = \nu\frac{\partial u}{\partial y}\Big|_{y=0}$$

Therefore

$$\int_0^\delta \frac{\partial}{\partial x}(u_\infty \cdot u - u \cdot u)\,dy = \nu\frac{\partial u}{\partial y}\Big|_{y=0}$$

Rearranging, we then obtain

$$\frac{d}{dx}\left[\int_0^\delta (u_\infty - u)u\,dy\right] = \nu\frac{\partial u}{\partial y}\Big|_{y=0} \tag{F.3}$$

Equation F.3 is the integral form of the boundary layer momentum equation. In a similar fashion, the following integral forms of the boundary layer energy and species continuity equations may be obtained:

$$\frac{d}{dx}\left[\int_0^{\delta_t} (T_\infty - T)u\,dy\right] = \alpha\frac{\partial T}{\partial y}\Big|_{y=0} \tag{F.4}$$

$$\frac{d}{dx}\left[\int_0^{\delta_c} (\rho_{A,\infty} - \rho_A)\, u\, dy\right] = D_{AB}\frac{\partial \rho_A}{\partial y}\bigg|_{y=0} \tag{F.5}$$

Equations F.3 through F.5 satisfy the x momentum, the energy, and the species conservation requirements in an *integral* (or *average*) fashion over the entire boundary layer. In contrast, the original conservation equations, (7.5) through (7.7), satisfy the conservation requirements *locally,* that is, at each point in the boundary layer.

The integral equations can be used to obtain *approximate* boundary layer solutions. The procedure involves first *assuming* reasonable functional forms for the unknowns u, T, and ρ_A in terms of the corresponding (*unknown*) boundary layer thicknesses. The assumed forms must satisfy appropriate boundary conditions. Substituting these forms into the integral equations, expressions for the boundary layer thicknesses may be determined and the assumed functional forms may then be completely specified. Although this method is approximate, it frequently leads to accurate results for the surface parameters.

Consider the hydrodynamic boundary layer, for which appropriate boundary conditions are

$$u(y=0) = \frac{\partial u}{\partial y}\bigg|_{y=\delta} = 0 \qquad \text{and} \qquad u(y=\delta) = u_\infty$$

From Equation 7.5 it also follows that, since $u = v = 0$ at $y = 0$,

$$\frac{\partial^2 u}{\partial y^2}\bigg|_{y=0} = 0$$

With the foregoing conditions, we could approximate the velocity profile as a third-degree polynomial of the form

$$\frac{u}{u_\infty} = a_1 + a_2\left(\frac{y}{\delta}\right) + a_3\left(\frac{y}{\delta}\right)^2 + a_4\left(\frac{y}{\delta}\right)^3$$

and apply the conditions to determine the coefficients a_1 to a_4. It is easily verified that $a_1 = a_3 = 0$, $a_2 = \frac{3}{2}$ and $a_4 = -\frac{1}{2}$, in which case

$$\frac{u}{u_\infty} = \frac{3}{2}\frac{y}{\delta} - \frac{1}{2}\left(\frac{y}{\delta}\right)^3 \tag{F.6}$$

The velocity profile is then specified in terms of the unknown boundary layer thickness δ. This unknown may be determined by substituting Equation F.6 into F.3 and integrating over y to obtain

$$\frac{d}{dx}\left(\frac{39}{280}u_\infty^2\,\delta\right) = \frac{3}{2}\frac{\nu u_\infty}{\delta}$$

Separating variables and integrating over x, we obtain

$$\frac{\delta^2}{2} = \frac{140}{13}\frac{\nu x}{u_\infty} + \text{constant}$$

However, since $\delta = 0$ at the leading edge of the plate ($x = 0$), the integration constant must be zero and

$$\delta = 4.64\left(\frac{\nu x}{u_\infty}\right)^{1/2} = \frac{4.64x}{Re_x^{1/2}} \tag{F.7}$$

Substituting Equation F.7 into Equation F.6 and evaluating $\tau_s = \mu(\partial u/\partial y)_s$, we also obtain

$$C_{f,x} = \frac{\tau_s}{\rho u_\infty^2/2} = \frac{0.646}{Re_x^{1/2}} \tag{F.8}$$

Despite the approximate nature of the foregoing procedure, Equations F.7 and F.8 compare quite well with results obtained from the exact solution, Equations 7.19 and 7.20.

In a similar fashion one could assume a temperature profile of the form

$$T^* = \frac{T - T_s}{T_\infty - T_s} = b_1 + b_2\left(\frac{y}{\delta_t}\right) + b_3\left(\frac{y}{\delta_t}\right)^2 + b_4\left(\frac{y}{\delta_t}\right)^3$$

and determine the coefficients from the conditions

$$T^*(y = 0) = \left.\frac{\partial T^*}{\partial y}\right|_{y=\delta_t} = 0$$

$$T^*(y = \delta_t) = 1$$

as well as

$$\left.\frac{\partial^2 T^*}{\partial y^2}\right|_{y=0} = 0$$

which is inferred from the energy equation (7.6). We then obtain

$$T^* = \frac{3}{2}\frac{y}{\delta_t} - \frac{1}{2}\left(\frac{y}{\delta_t}\right)^3 \tag{F.9}$$

Substituting Equations F.6 and F.9 into Equation F.4, we obtain, after some manipulation and assuming $Pr \gtrsim 1$,

$$\frac{\delta_t}{\delta} = \frac{Pr^{-1/3}}{1.026} \tag{F.10}$$

This result is in good agreement with that obtained from the exact solution, Equation 7.24. Moreover, the heat transfer coefficient may be then computed from

$$h = \frac{-k\,\partial T/\partial y|_{y=0}}{T_s - T_\infty} = \frac{3}{2}\frac{k}{\delta_t}$$

Substituting from Equations F.7 and F.10, we obtain

$$Nu_x = \frac{hx}{k} = 0.332 Re_x^{1/2} Pr^{1/3} \tag{F.11}$$

This result agrees precisely with that obtained from the exact solution, Equation 7.23. Using the same procedures, analogous results may be obtained for the concentration boundary layer.

References

1. von Kárman, T., *Z. Angew. Math. Mech.,* **1,** 232, 1921.
2. Pohlhausen, K., *Z. Angew. Math. Mech.,* **1,** 252, 1921.