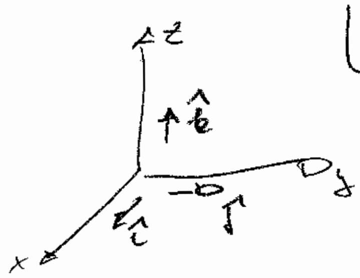
cc 1

Vettori



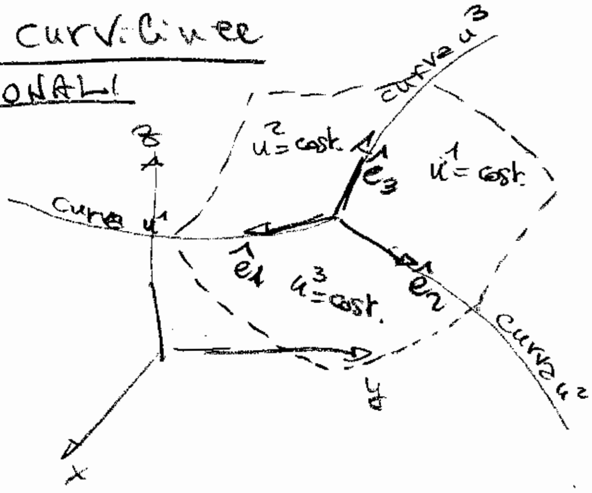
$$\bar r = x\,\hat i + y\,\hat j + z\,\hat k$$

nel sistema cartesiano ortogonale

$$\bar v = \dot{\bar r} = \dot x\,\hat i + \dot y\,\hat j + \dot z\,\hat k$$

$$\bar a = \ddot{\bar r} = \ddot x\,\hat i + \ddot y\,\hat j + \ddot z\,\hat k$$

Passiamo ora alle coordinate curvilinee ORTOGONALI



da $x, y, z \longrightarrow u^1, u^2, u^3$

$$ds^2 = dx^2 + dy^2 + dz^2 = g_{ij}\, du^i\, du^j$$

Ricordiamo che

$$\bar x_i \equiv \frac{\partial \bar r}{\partial u^i} = \left|\frac{\partial \bar r}{\partial u^i}\right| \cdot \hat e_i$$

con $\hat e_i$ versore tangente alla curva della coordinata u^i

Sarà
$$g_{ij} \equiv \bar x_i \cdot \bar x_j = \frac{\partial \bar r}{\partial u^i} \cdot \frac{\partial \bar r}{\partial u^j} = \left|\frac{\partial \bar r}{\partial u^i}\right| \left|\frac{\partial r}{\partial u^j}\right| \hat e_i \cdot \hat e_j$$

Se le coord. curvilinee sono a due a due ortogonali, $\hat e_i \cdot \hat e_j$ sarà nullo se $i \neq j$, per cui il tensor metrico g_{ij} risulterà avere elementi $\neq 0$ solo sulla diagonale principale, per cui

$$ds^2 = g_{11}(du^1)^2 + g_{22}(du^2)^2 + g_{33}(du^3)^2 = \underbrace{\underbrace{\left|\frac{\partial \bar r}{\partial u^1}\right|^2}_{h_1^2}(du^1)^2}_{ds_1^2} + \underbrace{\left|\frac{\partial \bar r}{\partial u^2}\right|^2}_{h_2^2}(du^2)^2 + \underbrace{\left|\frac{\partial \bar r}{\partial u^3}\right|^2}_{h_3^2}(du^3)^2$$

con $h_i \equiv \left|\frac{\partial \bar r}{\partial u^i}\right|$

ds_1^2 + analoghe

L'elemento di volume sarà

$$dV = \sqrt{g}\, du^1 du^2 du^3 = \sqrt{g_{11} \cdot g_{22} \cdot g_{33}}\, du^1 du^2 du^3 = \sqrt{h_1^2 h_2^2 h_3^2}\, du^1 du^2 du^3 = h_1 h_2 h_3\, du^1 du^2 du^3$$

Gradiente

$$\boxed{\overline{\nabla}\Phi = \frac{\partial \Phi}{\partial x}\hat{\imath} + \frac{\partial \Phi}{\partial y}\hat{\jmath} + \frac{\partial \Phi}{\partial z}\hat{k}}$$

Coord. cartesiane ortogonali

Il gradiente è il vettore con direzione e modulo della massima variazione spaziale della funzione Φ.
La componente del $\overline{\nabla}\phi$ nella direzione normale alla famiglia di superfici $u^1 =$ costante è dato dalla sua proiezione nella direzione del versore $\hat{e}_1$, $\perp$ alla sup. $u^1 =$ cost.

$$\hat{e}_1 \cdot \overline{\nabla}\Phi = \frac{\Delta \Phi}{\Delta s_1} = \frac{\Delta \Phi}{h_1 \Delta u^1} \xRightarrow{\lim} \frac{1}{h_1}\frac{\partial \phi}{\partial u^1}$$

nella direzione del versore $\hat{e}_1$, per cui la componente del gradiente sarà

$$\frac{1}{h_1}\frac{\partial \phi}{\partial u^1} \cdot \hat{e}_1$$

Nel complesso il gradiente di $\phi(u^i)$ sarà

$$\boxed{\overline{\nabla}\Phi = \frac{1}{h_i}\frac{\partial \phi}{\partial u^i}\hat{e}_i}$$

(somma su $i = 1, 2, 3$ vettoriale)

Divergenza

Useremo il teorema della divergenza (o di Gauss)

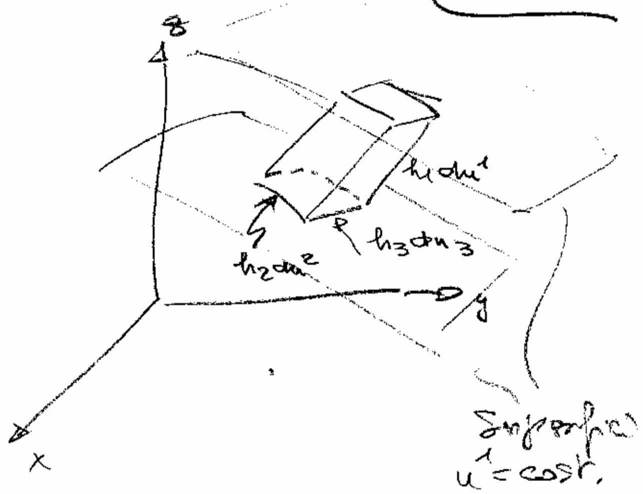
$$\int_{Vol.} \overline{\nabla} \cdot \overline{F}\, d\tau = \int_{Sup.} \overline{F} \cdot d\overline{\sigma}$$

che, per un volume $\int d\tau$ che tende a zero, si scrive anche

$$\overline{\nabla} \cdot \overline{F}(u^i) = \lim_{\int d\tau \to 0} \frac{\int \overline{F} \cdot d\overline{\sigma}}{\int d\tau}$$

in cui abbiamo visto che l'elemento di volume è $h_1 h_2 h_3\, du^1 du^2 du^3$.

Consideriamo un elemento di volume infinitesimo per valutare il flusso attraverso due facce $u^1 =$ costante del vettore $\overline{F}$. Chiamiamo F_1 la proiezione di $\overline{F}$ nella direzione della (= del versore $\vec{e}_1$) coord. u^1. Il flusso attraverso l'areola $h_2 du^2\, h_3 du^3$ corrispondente a un certo valore di u^1 sarà



$F_1 h_2 h_3\, du^2 du^3$.

Per avere il flusso devo fare la differenza tra i valori su $u^1 = cost.$ e $u^1 + du^1 = cost.$ (trascuro gli infinitesimi di ordine superiore al primo)

$$\left[F_1 h_2 h_3 + \frac{\partial}{\partial u^1}(F_1 h_2 h_3)du^1\right]du^2 du^3 - F_1 h_2 h_3\, du^2 du^3 =$$

$$= \frac{\partial}{\partial u^1}(F_1 h_2 h_3)\, du^1 du^2 du^3$$

Aggiungo i risultati analoghi per le altre due coppie di superfici e ottengo

$$\int \overline{F}\cdot d\overline{\sigma} = \left[\frac{\partial}{\partial u^1}(F_1 h_2 h_3) + \frac{\partial}{\partial u^2}(F_2 h_3 h_1) + \frac{\partial}{\partial u^3}(F_3 h_2 h_1)\right] du^1 du^2 du^3$$

divido per l'elemento di volume $h_1 h_2 h_3\, du^1 du^2 du^3$ e ottengo:

$$\boxed{\overline{\nabla}\cdot\overline{F}(u^i) = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial u^1}(F_1 h_2 h_3) + \frac{\partial}{\partial u^2}(F_2 h_3 h_1) + \frac{\partial}{\partial u^3}(F_3 h_2 h_1)\right]}$$

Poiché il laplaciano $\nabla^2\Phi$ è la divergenza del gradiente, e le componenti del $\nabla\Phi$ nella direzione $\vec{e}_i$ sono $\frac{1}{h_i}\frac{\partial \Phi}{\partial u^i}$, ottengo:

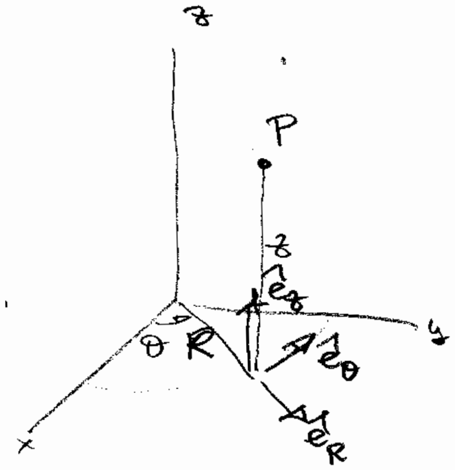
$$\boxed{\nabla^2\Phi(u^i) = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial u^1}\left(\frac{h_2 h_3}{h_1}\frac{\partial \Phi}{\partial u^1}\right) + \frac{\partial}{\partial u^2}\left(\frac{h_3 h_1}{h_2}\frac{\partial \Phi}{\partial u^2}\right) + \frac{\partial}{\partial u^3}\left(\frac{h_2 h_1}{h_3}\frac{\partial \Phi}{\partial u^3}\right)\right]}$$

Casi particolari

① Coordinate cartesiane ortogonali

In questo caso $h_1 = h_2 = h_3 \equiv 1$ e con le formule scritte sopra riotteniamo banalmente le formule classiche.

② Coordinate cilindriche (R, θ, z)



$$\begin{cases} x = R\cos\theta \\ y = R\,\text{sen}\,\theta \\ z = z \end{cases} \rightarrow \overline{r} \equiv (R\cos\theta, R\,\text{sen}\,\theta, z)$$

$$\overline{X}_R = \frac{\partial \overline{r}}{\partial R} = (\cos\theta, \text{sen}\,\theta, 0)$$

$$\overline{X}_\theta = \frac{\partial \overline{r}}{\partial \theta} = (-R\,\text{sen}\,\theta, R\cos\theta, 0)$$

$$\overline{X}_t = \frac{\partial \overline{r}}{\partial z} = (0, 0, 1)$$

$$g_{RR} = \overline{X}_R \cdot \overline{X}_R = \cos^2\theta + \text{sen}^2\theta = 1 \rightarrow g_{RR} = h_R^2 \rightarrow \boxed{h_R \equiv 1}$$

$$g_{\theta\theta} = \overline{X}_\theta \cdot \overline{X}_\theta = R^2\,\text{sen}^2\theta + R^2\cos^2\theta = R^2 \rightarrow g_{\theta\theta} = h_\theta^2 \Rightarrow \boxed{h_\theta = R}$$

$$g_{zz} = \overline{X}_z \cdot \overline{X}_z = 1 \rightarrow g_{zz} = h_z^2 \rightarrow \boxed{h_z = 1}$$

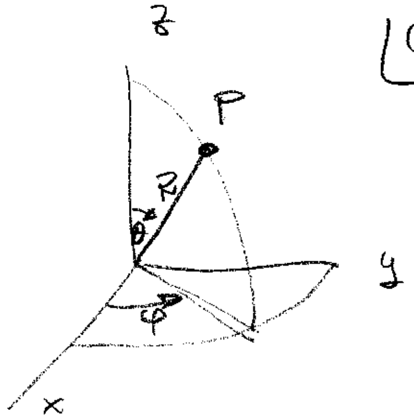
- $dV = R\, dR\, d\theta\, d\varphi$

- $$\overline{\nabla}\Phi(R,\theta,z) = \frac{\partial \Phi}{\partial R}\hat{e}_R + \frac{1}{R}\frac{\partial \Phi}{\partial \theta}\hat{e}_\theta + \frac{\partial \Phi}{\partial z}\hat{e}_z$$

- $$\overline{\nabla} \cdot \overline{F}(R,\theta,z) = \frac{1}{R}\left[\frac{\partial}{\partial R}(F_R \cdot R) + \frac{\partial F_\theta}{\partial \theta} + \frac{\partial}{\partial z}(F_z \cdot R)\right] = \frac{1}{R}\frac{\partial (F_R \cdot R)}{\partial R} + \frac{1}{R}\frac{\partial F_\theta}{\partial \theta} + \frac{\partial F_z}{\partial z}$$

- $$\nabla^2\Phi(R,\theta,z) = \frac{1}{R}\left[\frac{\partial}{\partial R}\left(R\frac{\partial \Phi}{\partial R}\right) + \frac{\partial}{\partial \theta}\left(\frac{1}{R}\frac{\partial \Phi}{\partial \theta}\right) + \frac{\partial}{\partial z}\left(R\frac{\partial \Phi}{\partial z}\right)\right] = \frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial \Phi}{\partial R}\right) + \frac{1}{R^2}\frac{\partial^2 \Phi}{\partial \theta^2} + \frac{\partial^2 \Phi}{\partial z^2}$$

③ Coordinate sferiche (R, θ, φ) CC5



$$\begin{cases} x = R \operatorname{sen}\theta \cos\varphi \\ y = R \operatorname{sen}\theta \operatorname{sen}\varphi \\ z = R \cos\theta \end{cases}$$

$$\overline{r} = \left(R \operatorname{sen}\theta \cos\varphi, R \operatorname{sen}\theta \operatorname{sen}\varphi, R\cos\theta\right)$$

$$\overline{x}_R = \frac{\partial \overline{r}}{\partial R} = (\operatorname{sen}\theta\cos\varphi, \operatorname{sen}\theta\operatorname{sen}\varphi, \cos\theta)$$

$$\overline{x}_\theta = \frac{\partial \overline{r}}{\partial \theta} = (R\cos\theta\cos\varphi, R\cos\theta\operatorname{sen}\varphi, -R\operatorname{sen}\theta)$$

$$\overline{x}_\varphi = \frac{\partial \overline{r}}{\partial \varphi} = (-R\operatorname{sen}\theta\operatorname{sen}\varphi, R\operatorname{sen}\theta\cos\varphi, 0)$$

$$g_{RR} = \overline{x}_R \cdot \overline{x}_R = \operatorname{sen}^2\theta\cos^2\varphi + \operatorname{sen}^2\theta\operatorname{sen}^2\varphi + \cos^2\theta = 1 \rightarrow \boxed{h_R = \sqrt{g_{RR}} = 1}$$

$$g_{\theta\theta} = \overline{x}_\theta \cdot \overline{x}_\theta = R^2\cos^2\theta\cos^2\varphi + R^2\cos^2\theta\operatorname{sen}^2\varphi + R^2\operatorname{sen}^2\theta = R^2 \rightarrow \boxed{h_\theta = \sqrt{g_{\theta\theta}} = R}$$

$$g_{\varphi\varphi} = \overline{x}_\varphi \cdot \overline{x}_\varphi = R^2\operatorname{sen}^2\theta\operatorname{sen}^2\varphi + R^2\operatorname{sen}^2\theta\cos^2\varphi = R^2\operatorname{sen}^2\theta \rightarrow \boxed{h_\varphi = \sqrt{g_{\varphi\varphi}} = R\operatorname{sen}\theta}$$

- $dV = R^2 \operatorname{sen}\theta \, d\theta \, d\varphi$

- $$\overline{\nabla}\phi(R,\theta,\varphi) = \frac{\partial\Phi}{\partial R}\hat{e}_R + \frac{1}{R}\frac{\partial\Phi}{\partial\theta}\hat{e}_\theta + \frac{1}{R\operatorname{sen}\theta}\frac{\partial\Phi}{\partial\varphi}\hat{e}_\varphi$$

- $$\overline{\nabla}\cdot\overline{F}(R,\theta,\varphi) = \frac{1}{R^2\operatorname{sen}\theta}\left[\frac{\partial}{\partial R}(R^2\operatorname{sen}\theta\, F_R) + \frac{\partial}{\partial\theta}(R\operatorname{sen}\theta\, F_\theta) + \frac{\partial}{\partial\varphi}(R\, F_\varphi)\right]$$

$$= \frac{1}{R^2}\frac{\partial}{\partial R}(R^2 F_R) + \frac{1}{R\operatorname{sen}\theta}\frac{\partial}{\partial\theta}(\operatorname{sen}\theta\cdot F_\theta) + \frac{1}{R\operatorname{sen}\theta}\frac{\partial F_\varphi}{\partial\varphi}$$

- $$\nabla^2\Phi(R,\theta,\varphi) = \frac{1}{R^2\operatorname{sen}\theta}\left[\frac{\partial}{\partial R}\left(R^2\operatorname{sen}\theta\frac{\partial\Phi}{\partial R}\right) + \frac{\partial}{\partial\theta}\left(\operatorname{sen}\theta\frac{\partial\Phi}{\partial\theta}\right) + \frac{\partial}{\partial\varphi}\left(\frac{1}{\operatorname{sen}\theta}\frac{\partial\Phi}{\partial\varphi}\right)\right]$$

$$= \frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2\cdot\frac{\partial\phi}{\partial R}\right) + \frac{1}{R^2\operatorname{sen}\theta}\frac{\partial}{\partial\theta}\left(\operatorname{sen}\theta\frac{\partial\Phi}{\partial\theta}\right) + \frac{1}{R^2\operatorname{sen}^2\theta}\frac{\partial^2\Phi}{\partial\varphi^2}$$

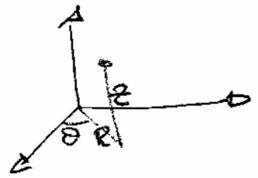
$\boxed{\text{Velocità}}$ $\quad \bar v = \dot{\bar r}$

$$\boxed{\dot{\bar r} = \frac{d\bar r}{dt} = \frac{\partial \bar r}{\partial u^i}\frac{du^i}{dt} = \underbrace{\left|\frac{\partial \bar r}{\partial u^i}\right|}_{h_i}\hat e_i \cdot \frac{du^i}{dt} = h_i \frac{du^i}{dt}\hat e_i}$$

Coord. cilindriche $\quad u^1 = R \quad u^2 = \vartheta \quad u^3 = z$

$h_R \equiv 1 \qquad h_\vartheta \equiv R \qquad h_z \equiv 1$

$$\dot{\bar r} = \bar v = \underbrace{\dot R}_{v_R}\hat e_R + \underbrace{R\dot\vartheta}_{v_\vartheta}\hat e_\vartheta + \underbrace{\dot z}_{v_z}\hat e_z$$

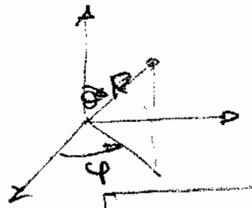


$\boxed{\dot R = v_R} \qquad R\dot\vartheta = v_\vartheta \to \boxed{\dot\vartheta = v_\vartheta / R} \qquad \boxed{\dot z = v_z}$

Coord. sferiche $\quad u^1 = R \quad u^2 = \vartheta \quad u^3 = \varphi$

$h_R = 1 \qquad h_\vartheta = R \qquad h_\varphi = R\,\text{sen}\,\vartheta$

$$\dot{\bar r} = \bar v = \underbrace{\dot R}_{v_R}\hat e_R + \underbrace{R\dot\vartheta}_{v_\vartheta}\hat e_\vartheta + \underbrace{R\,\text{sen}\,\vartheta\,\dot\varphi}_{v_\varphi}\hat e_\varphi$$



$\boxed{\dot R = v_R} \qquad R\dot\vartheta = v_\vartheta \to \boxed{\dot\vartheta = v_\vartheta / R} \qquad R\,\text{sen}\,\vartheta\,\dot\varphi = v_\varphi \to \boxed{\dot\varphi = \dfrac{v_\varphi}{R\,\text{sen}\,\vartheta}}$

Da queste relazioni otteniamo le derivate delle velocità:

Coord. cilindriche

$$\begin{cases} v_R = \dot R \\ v_\vartheta = R\dot\vartheta \\ v_z = \dot z \end{cases} \Rightarrow \begin{cases} \dot v_R = \ddot R \\ \dot v_\vartheta = \dot R\dot\vartheta + R\ddot\vartheta \\ \dot v_z = \ddot z \end{cases}$$

Coord. sferiche

$$\begin{cases} v_R = \dot R \\ v_\vartheta = R\dot\vartheta \\ v_\varphi = R\,\text{sen}\,\vartheta\,\dot\varphi \end{cases} \Rightarrow \begin{cases} \dot v_R = \ddot R \\ \dot v_\vartheta = \dot R\dot\vartheta + R\ddot\vartheta \\ \dot v_\varphi = \dot R\,\text{sen}\,\vartheta\,\dot\varphi + R\cos\vartheta\,\dot\vartheta\dot\varphi + \\ \quad + R\,\text{sen}\,\vartheta\,\ddot\varphi \end{cases}$$

Accelerazione

$$\dot{\bar r} = h_i \frac{du^i}{dt}\hat e_i \Rightarrow \ddot{\bar r} = \frac{d}{dt}\left[h_i \frac{du^i}{dt}\hat e_i\right] =$$

$$\ddot{\bar r} = \dot h_i \frac{du^i}{dt}\hat e_i + h_i\left[\frac{d^2u^i}{dt^2}\hat e_i + \frac{du^i}{dt}\dot{\hat e}_i\right] =$$

$$\boxed{\ddot{\bar r} = \left[\dot h_i \frac{du^i}{dt} + h_i\frac{d^2u^i}{dt^2}\right]\hat e_i + h_i\frac{du^i}{dt}\dot{\hat e}_i}$$

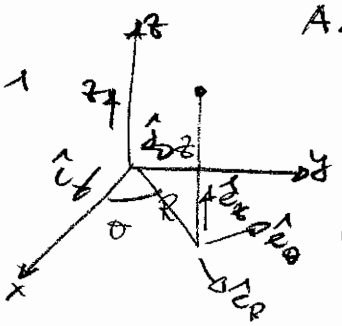
Gli $\dot{\hat e}_i$ dipendono dal sistema di riferimento

Coord. cilindriche

$u^1=R \quad u^2=\theta \quad u^3=z$

$h_R=1 \quad h_\theta=R \quad h_z=1$

$$\bar A\times\bar B = \begin{vmatrix} \hat i & \hat j & \hat k\\ A_x & A_y & A_z \\ B_x & B_y & B_z\end{vmatrix}$$



$$\hat e_\theta = \hat e_z \times \hat e_R$$

$\hat e_R = \hat i\cos\theta + \hat j\sin\theta$ (NB: modulo = 1)

$\hat e_z = \hat k$

$$\hat e_\theta = \begin{vmatrix} \hat i & \hat j & \hat k\\ 0 & 0 & 1 \\ \cos\theta & \sin\theta & 0\end{vmatrix} = -\hat i\sin\theta + \hat j\cos\theta$$

$\hat e_R$ e $\hat e_\theta$ dipendono dall'angolo θ, $\hat e_z$ nemmeno da quello, per cui tra le derivate $\frac{\partial \hat e_i}{\partial u^j}$ sono $\neq 0$ solo

$$\begin{cases} \dfrac{\partial \hat e_R}{\partial\theta} = -\hat i\sin\theta + \hat j\cos\theta \equiv \hat e_\theta \\[2ex] \dfrac{\partial \hat e_\theta}{\partial\theta} = -\hat i\cos\theta - \hat j\sin\theta \equiv -\hat e_R \end{cases}$$

Perciò $\dot{\hat e}_i \equiv \frac{\partial \hat e_i}{\partial u^j}\cdot \dot u^j \Rightarrow \begin{cases} \dot{\hat e}_R = \dot\theta\,\hat e_\theta \\ \dot{\hat e}_\theta = -\dot\theta\,\hat e_R \end{cases}$

$+R\dot\theta\dot{\hat e}_\theta =$
$= R\dot\theta(-\dot\theta\,\hat e_R)$

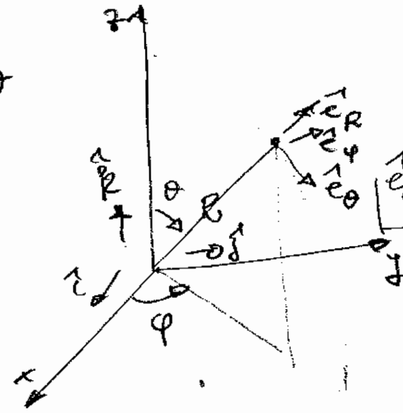
A questo punto possiamo scrivere:

$$\ddot{\bar r} = \ddot R\hat e_R + \dot R\underbrace{\dot\theta\hat e_\theta}_{\dot{\hat e}_R} + [\dot R\dot\theta + R\ddot\theta]\hat e_\theta - R\dot\theta^2\hat e_R + \ddot z\hat e_z$$

$$\boxed{\ddot{\bar r} = (\ddot R - R\dot\theta^2)\hat e_R + (2\dot R\dot\theta + R\ddot\theta)\hat e_\theta + \ddot z\hat e_z}$$

$\boxed{\text{Coord. sferiche}}$ $u^1 = R \quad u^2 = \theta \quad u^3 = \varphi$

$h_R = 1 \quad h_\theta = R \quad h_\varphi = R\,\mathrm{sen}\,\theta$



$\hat{e}_\theta = \hat{e}_\varphi \times \hat{e}_R$

$$\begin{cases} \hat{e}_R = \hat{\imath}\,\mathrm{sen}\theta\cos\varphi + \hat{\jmath}\,\mathrm{sen}\theta\,\mathrm{sen}\varphi + \hat{k}\cos\theta \\ \hat{e}_\varphi = \dfrac{\hat{k}\times\hat{e}_R}{\mathrm{sen}\theta} = \dfrac{1}{\mathrm{sen}\theta}\cdot\left[-\hat{\imath}\,\mathrm{sen}\theta\,\mathrm{sen}\varphi + \hat{\jmath}\,\mathrm{sen}\theta\cos\varphi\right] \\ \quad = -\hat{\imath}\,\mathrm{sen}\varphi + \hat{\jmath}\cos\varphi \\ \hat{e}_\theta = \hat{\imath}\cos\varphi\cos\theta + \hat{\jmath}\,\mathrm{sen}\varphi\cos\theta - \hat{k}\,\mathrm{sen}\theta \end{cases}$$

$$\hat{k}\times\hat{e}_R \equiv \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 0 & 0 & 1 \\ \mathrm{sen}\theta\cos\varphi & \mathrm{sen}\theta\,\mathrm{sen}\varphi & \cos\theta \end{vmatrix}$$

$$\hat{e}_\varphi\times\hat{e}_R = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -\mathrm{sen}\varphi & \cos\varphi & 0 \\ \mathrm{sen}\theta\cos\varphi & \mathrm{sen}\theta\,\mathrm{sen}\varphi & \cos\theta \end{vmatrix}$$

$\hat{e}_R$ e $\hat{e}_\theta$ dipendono da θ e φ, $\hat{e}_\varphi$ solo da φ

Saranno quindi non nulle solo le seguenti derivate:

$$\begin{cases} \dfrac{\partial\hat{e}_R}{\partial\theta} = \hat{\imath}\cos\theta\cos\varphi + \hat{\jmath}\cos\theta\,\mathrm{sen}\varphi - \hat{k}\,\mathrm{sen}\theta \equiv \hat{e}_\theta \\ \dfrac{\partial\hat{e}_R}{\partial\varphi} = -\hat{\imath}\,\mathrm{sen}\theta\,\mathrm{sen}\varphi + \hat{\jmath}\,\mathrm{sen}\theta\cos\varphi = \mathrm{sen}\theta\cdot\hat{e}_\varphi \\ \dfrac{\partial\hat{e}_\varphi}{\partial\varphi} = -\hat{\imath}\cos\varphi - \hat{\jmath}\,\mathrm{sen}\varphi = -(\hat{e}_R\,\mathrm{sen}\theta + \hat{e}_\theta\cos\theta) \\ \dfrac{\partial\hat{e}_\theta}{\partial\theta} = -\hat{\imath}\cos\varphi\,\mathrm{sen}\theta - \hat{\jmath}\,\mathrm{sen}\varphi\,\mathrm{sen}\theta - \hat{k}\cos\theta = -\hat{e}_R \\ \dfrac{\partial\hat{e}_\theta}{\partial\varphi} = -\hat{\imath}\,\mathrm{sen}\varphi\cos\theta + \hat{\jmath}\cos\varphi\cos\theta = \cos\theta\cdot\hat{e}_\varphi \end{cases}$$

Perciò sarà

$$\dot{\hat{e}}_i \equiv \frac{\partial\hat{e}_i}{\partial u^j}\dot{u}^j \Longrightarrow \begin{cases} \dot{\hat{e}}_R = \dot{\theta}\,\hat{e}_\theta + \mathrm{sen}\theta\,\dot{\varphi}\,\hat{e}_\varphi \\ \dot{\hat{e}}_\varphi = -\dot{\varphi}\left(\hat{e}_R\,\mathrm{sen}\theta + \hat{e}_\theta\cos\theta\right) \\ \dot{\hat{e}}_\theta = -\dot{\theta}\,\hat{e}_R + \cos\theta\,\dot{\varphi}\,\hat{e}_\varphi \end{cases}$$

Scriveremo:

$$\ddot{\vec{r}} = \ddot{R}\,\hat{e}_R + \dot{R}\dot{\theta}\,\hat{e}_\theta + \dot{R}\,\mathrm{sen}\theta\,\dot{\varphi}\,\hat{e}_\varphi + [\dot{R}\dot{\theta} + R\ddot{\theta}]\,\hat{e}_\theta + R\dot{\theta}\,[-\dot{\theta}\,\hat{e}_R + \cos\theta\,\dot{\varphi}\,\hat{e}_\varphi] +$$
$$+ [\dot{R}\,\mathrm{sen}\theta\,\dot{\varphi} + R\cos\theta\,\dot{\theta}\,\dot{\varphi} + R\,\mathrm{sen}\theta\,\ddot{\varphi}]\,\hat{e}_\varphi + R\,\mathrm{sen}\theta\,\dot{\varphi}\,[-\dot{\varphi}\,\mathrm{sen}\theta\,\hat{e}_R - \dot{\varphi}\cos\theta\,\hat{e}_\theta] =$$
$$= \hat{e}_R(\ddot{R} - R\dot{\theta}^2 - R\,\mathrm{sen}^2\theta\,\dot{\varphi}^2] + \hat{e}_\theta[2\dot{R}\dot{\theta} + R\ddot{\theta} - R\,\mathrm{sen}\theta\cos\theta\,\dot{\varphi}^2] +$$
$$+ \hat{e}_\varphi[2\dot{R}\,\mathrm{sen}\theta\,\dot{\varphi} + 2R\dot{\theta}\cos\theta\,\dot{\varphi} + R\,\mathrm{sen}\theta\,\ddot{\varphi}]$$

Se ci troviamo in un sistema a simmetria sferica la forza agirà solamente lungo $\hat{e}_R$ e sarà

$$\overline{F} = -m \frac{d\phi}{dR}\hat{e}_R = m\overline{a} \rightarrow \boxed{\overline{a} = \ddot{\overline{r}} = -\frac{d\phi}{dR}\cdot\hat{e}_R}$$

mentre saranno nulle le componenti di $\overline{a}$ lungo $\hat{e}_\theta$ ed $\hat{e}_\varphi$.

Dalla relazione appena ricavata per $\ddot{\overline{r}}$ avremo allora:

- componente $\hat{e}_R$?

$$\ddot{R} - R\dot{\theta}^2 - R\,\mathrm{sen}^2\theta\,\dot{\varphi}^2 \equiv -\frac{d\phi}{dR}$$

Ricordiamo [CC6]

$$\dot{R} = v_R \quad \dot{\theta} = v_\theta/R \quad \dot{\varphi} = \frac{v_\varphi}{R\,\mathrm{sen}\theta}$$

$$\dot{v}_R = \ddot{R}$$

$$\dot{v}_\theta = \dot{R}\dot{\theta} + R\ddot{\theta}$$

$$\dot{v}_\varphi = \dot{R}\,\mathrm{sen}\theta\,\dot{\varphi} + R\cos\theta\,\dot{\theta}\dot{\varphi} + R\,\mathrm{sen}\theta\,\ddot{\varphi}$$

dalla $v_R = \dot{R} \rightarrow \dot{v}_R = \ddot{R}$ cioè

$$\dot{v}_R = \ddot{R} = -\frac{d\phi}{dR} + R\dot{\theta}^2 + R\,\mathrm{sen}^2\theta\,\dot{\varphi}^2$$

$$= -\frac{d\phi}{dR} + \cancel{R}\cdot\frac{v_\theta^2}{R^{\cancel{2}}} + \cancel{R}\,\cancel{\mathrm{sen}^2\theta}\,\frac{v_\varphi^2}{R^{\cancel{2}}\,\cancel{\mathrm{sen}^2\theta}}$$

$$\boxed{\dot{v}_R = -\frac{d\phi}{dR} + \frac{v_\theta^2 + v_\varphi^2}{R}}$$

- componente $\hat{e}_\theta = 0$

$$2\dot{R}\dot{\theta} + R\ddot{\theta} - R\,\mathrm{sen}\theta\cos\theta\,\dot{\varphi}^2 \equiv 0$$

$$\boxed{\dot{v}_\theta = \dot{R}\dot{\theta} + R\ddot{\theta} = \cancel{R}\,\cancel{\mathrm{sen}\theta}\cos\theta\cdot\frac{v_\varphi^2}{R^{\cancel{2}}\,\mathrm{sen}^{\cancel{2}}\theta} - v_R\cdot\frac{v_\theta}{R} = \cot\theta\,\frac{v_\varphi^2}{R} - \frac{v_R v_\theta}{R}}$$

- componente $\hat{e}_\varphi = 0$

$$2\dot{R}\,\mathrm{sen}\theta\,\dot{\varphi} + 2R\dot{\theta}\cos\theta\,\dot{\varphi} + R\,\mathrm{sen}\theta\,\ddot{\varphi} \equiv 0$$

$$\dot{v}_\varphi = \dot{R}\,\mathrm{sen}\theta\,\dot{\varphi} + R\cos\theta\,\dot{\theta}\dot{\varphi} + R\,\mathrm{sen}\theta\,\ddot{\varphi} = -\dot{R}\,\mathrm{sen}\theta\,\dot{\varphi} - R\cos\theta\,\dot{\theta}\dot{\varphi} =$$

$$= -v_R\,\cancel{\mathrm{sen}\theta}\,\frac{v_\varphi}{R\,\cancel{\mathrm{sen}\theta}} - \cancel{R}\cos\theta\,\frac{v_\theta}{\cancel{R}}\cdot\frac{v_\varphi}{R\,\mathrm{sen}\theta} =$$

$$\boxed{\dot{v}_\varphi = -\frac{v_R v_\varphi + \cot\theta\, v_\theta v_\varphi}{R}}$$

Se ho una funzione $f = f(R, \theta, \varphi, v_R, v_\theta, v_\varphi)$ e voglio fare la sua $\frac{df}{dt}$, sarà

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \dot{R}\frac{\partial f}{\partial R} + \dot{\theta}\frac{\partial f}{\partial \theta} + \dot{\varphi}\frac{\partial f}{\partial \varphi} + \dot{v}_R\frac{\partial f}{\partial v_R} + \dot{v}_\theta\frac{\partial f}{\partial v_\theta} + \dot{v}_\varphi\frac{\partial f}{\partial v_\varphi} =$$

$$= \frac{\partial f}{\partial t} + v_R\frac{\partial f}{\partial R} + \frac{v_\theta}{R}\frac{\partial f}{\partial \theta} + \frac{v_\varphi}{R\,\mathrm{sen}\theta}\frac{\partial f}{\partial \varphi} + \left(\frac{v_\theta^2 + v_\varphi^2}{R} - \frac{d\phi}{dR}\right)\frac{\partial f}{\partial v_R} + \frac{\cot\theta\, v_\varphi^2 - v_R v_\theta}{R}\frac{\partial f}{\partial v_\theta} +$$

$$- \frac{v_R v_\varphi + \cot\theta\, v_\theta v_\varphi}{R}\frac{\partial f}{\partial v_\varphi}$$

Come si vedrà, $\frac{df}{dt} = 0$ è l'eq. di Boltzmann non collisionale.