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$$\begin{cases} u_t = u_{xx} & \text{in } \mathbb{R}\times\mathbb{R}^+ \\ u(x,0) = f(x) & \text{in } \mathbb{R} \end{cases}$$

$$\hat{u}(\xi,t) = \int e^{-2\pi i \xi x}\, u(x,t)\, dx$$

$$-(2\pi\xi)^2\, \hat{u}(\xi,t) = \int \frac{d^2}{dx^2}\left(e^{-2\pi i \xi x}\right) u(x,t)\, dx =$$

$$= \int e^{-2\pi i \xi x}\, u_{xx}(x,t)\, dx$$

Quindi

$$\begin{cases} \frac{\partial}{\partial t}\hat{u} + (2\pi\xi)^2 \hat{u} = 0 \\ \hat{u}(\xi,0) = \hat{f} \end{cases}$$

$$\hat{u}(\xi,t) = e^{-(2\pi\xi)^2 t}\, \hat{f}(\xi)$$

$$u(x,t) = \int e^{2\pi i x\xi - (2\pi\xi)^2 t}\, \hat{f}(\xi)\, d\xi$$

$$= K(\cdot,t) * f\,(x)$$

$$K(x,t) = \int e^{2\pi i x\xi - (2\pi\xi)^2 t}\, d\xi =$$

$$= \mathcal{F}\left(e^{-(2\pi\xi)^2 t}\right)(x)$$

Ricordiamo $G(x) = e^{-\pi x^2}$

$$\hat{G}(\xi) = e^{-\pi \xi^2}$$

$$K(x,t) := \int e^{2\pi i x \xi} \, e^{-(2\pi \xi)^2 t} \, d\xi =$$

$$= \int e^{2\pi i x \xi} \, e^{-(4\pi t)\pi \xi^2} \, d\xi$$

$$y = \sqrt{4\pi t}\,\xi \qquad d\xi = \frac{1}{\sqrt{4\pi t}}\,dy$$

$$x\xi = \frac{x}{\sqrt{4\pi t}}\,y$$

$$K(x,t) = \int e^{2\pi i \left(\frac{x}{\sqrt{4\pi t}}\right) y} \, \frac{G(y)}{\sqrt{4\pi t}} \, dy =$$

$$= \frac{1}{\sqrt{4\pi t}}\, G\left(\frac{x}{\sqrt{\ }}\right)$$

$$= \frac{1}{\sqrt{4\pi t}}\, e^{-\left(\frac{\cancel{\pi} x^2}{4\cancel{\pi} t}\right)}$$

Avevamo posto $G_s(x) = \frac{1}{s} G\left(\frac{x}{s}\right)$

$$K(x,t) = G_s(x) \quad , \; s = \sqrt{4\pi t}$$

Quindi $K(\cdot,t)$ è una famiglia di buoni nuclei per $t \searrow 0$.

NB. $K_{xx} - K_t = 0$ su $\mathbb{R} \times (0,\infty)$.

quindi se $f \in \mathcal{S}(\mathbb{R})$

$$u = K(\cdot,t) * f(x) \text{ è soluz.}$$

dell' eq. del calore e

$$u(x,t) \to f(x) \text{ per } t \to 0^+.$$

Vediamo che questa formula è valida con più ampia generalità!

$$(x-\xi)^2 = x^2 - 2x\xi + \xi^2$$

$$|2x\xi| \leq \varepsilon^{-1}x^2 + \frac{1}{\varepsilon}\xi^2 \qquad \forall\, \varepsilon > 0.$$

$$-(x-\xi)^2 \geq -\left(1+\frac{1}{\varepsilon}\right)x^2 - (1+\varepsilon)\xi^2$$

$$-(x-\xi)^2 \leq -\left(1-\frac{1}{\varepsilon}\right)x^2 - (1-\varepsilon)\xi^2$$

$$\exp\left\{-\frac{\left(1+\frac{1}{\varepsilon}\right)}{4t}x^2\right\}\exp\left\{-\frac{(1+\varepsilon)}{4t}\xi^2\right\} \leq$$

$$\leq \exp\left\{-\frac{(x-\xi)^2}{4t}\right\} \leq$$

$$\exp\left\{-\frac{\left(1-\frac{1}{\varepsilon}\right)}{4t}x^2\right\}\exp\left\{-\frac{(1-\varepsilon)}{4t}\xi^2\right\}$$

$$F_1(x,t)\exp\left\{-\frac{(1+\varepsilon)\xi^2}{4t}\right\} \leq K(x-\xi,t) \leq F_2(x,t)\exp\left\{-\frac{(1-\varepsilon)}{4t}\xi^2\right\}$$

$$F_1(x,t) = \frac{1}{\sqrt{4\pi t}}\exp\left\{-\frac{\left(1+\frac{1}{\varepsilon}\right)}{4t}x^2\right\}$$

$$F_2(x,t) = \frac{1}{\sqrt{4\pi t}}\exp\left\{-\frac{\left(1-\frac{1}{\varepsilon}\right)}{4t}x^2\right\}.$$

Supp.

$$|f(x)| \leq C_1 \exp\{C_2 x^2\}$$

$$|f(\xi)\, \Gamma(x-\xi, t)| \leq C_1 F_\varepsilon(x,t) \cdot$$

$$\cdot \exp\left\{ C_2 \xi^2 - \frac{(1-\varepsilon)}{4t} \xi^2 \right\}$$

Se $0 < \varepsilon < 1$ e $\qquad 0 < t < \dfrac{1-\varepsilon}{4C_2}$

$$| \quad " \quad | \leq C_1 F_\varepsilon(x,t) \underbrace{\exp\left\{\left(C_2 - \frac{(1-\varepsilon)}{4t}\right)\xi^2\right\}}_{\text{integrabile !}}$$

$$|u(x,t)| \leq \text{Cost. } F_\varepsilon(x,t) \leq$$

$$\leq \text{Cost } \frac{1}{\sqrt{4\pi t}} \exp\left\{ - \frac{\left(1-\frac{1}{\varepsilon}\right)}{4t} x^2 \right\}$$

$$\leq \text{Cost. } \frac{1}{\sqrt{4\pi t}} \exp\left\{ \frac{\frac{1}{\varepsilon}-1}{1-\varepsilon} 4C_2 x^2 \right\}$$

u è def in $\mathbb{R} \times \left(0, \dfrac{1-\varepsilon}{4C_2}\right)$ ed ha una crescita del tipo di f.

<u>Teor.</u> Sia f continua t.c.

$$|f(x)| \le C_1 e^{C_2 x^2}$$

con $C_1, C_2 > 0$.

Allora $\lim_{t \searrow 0} u(x,t) = f(x)$

<u>Dim.</u>

$$u(x,t) - f(x) = \int K(x-\xi, t)\,(f(\xi) - f(x)) =$$

$$= \int_{|x-\xi|<\delta} + \int_{|x-\xi|>\delta} \quad \cdots$$

$\square$

<u>Teor di esistenza</u>

Se $|f(x)| \le C_1 e^{C_2 x^2}$

allora esiste soluz. $u(x,t)$ in

$$\mathbb{R} \times \left(0, \frac{1}{4C_2}\right).$$

Se $|f(x)| \le C_1 e^{C_2 (x)^{1+\alpha}}$, $0 \le \alpha < 1$

soluz esiste per tutti i tempi

<u>Unicità</u>.

Sia u soluz. t.c. per tutti i tempi si maggiora

$$|u(x,t)| \le C_1 e^{C_2 x^2}$$

con $C_1, C_2 > 0$. Allora il pb. ai val. iniz. ha soluz unica.

Consid v t.c.

$$\begin{cases} v_t - v_{xx} = 0 & \text{in } \mathbb{R}\times(0,T) \\ v(x,0) = 0 \\ |v(x,t)| \le C_1 \exp\{C_2 x^2\}. \end{cases}$$

Se proviamo che $v \equiv 0$ si prova l'unicità per tutti i tempi (basta riprendere come valore iniziale quello a $t = T-\tau$, $\tau > 0$ piccolo e si va avanti fino a $2T-\tau$ traslando $t \to (t - T + \tau)$).

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Sia $c_3 > 0$ t.c. $c_3 > c_2$, $c_3 < \frac{1}{4T}$.

$$W(x,t) = \frac{1}{(1-4c_3t)^{1/2}} \exp\left\{ c_3 \frac{x^2}{(1-4c_3t)} \right\}$$

si verifica che
$$W_t - W_{xx} = 0. \qquad 0 < t < T$$

$$W(x,0) = \exp\{c_3 x^2\}.$$

Si fissa (x_1, t_1), $t_1 < \frac{1}{4c_3}$

si prende $A > |x_1|$ e si consid.

$$R = [-A, A] \times [0, \tfrac{1}{4c_3})$$

$$z(x,t) = c_1 \exp\{(c_2 - c_3) A^2\} \, W(x,t).$$

$$|v(x,0)| = 0 < z(x,0)$$

$$|v(\pm A, t)| \leq c_1 e^{c_2 A^2}$$

$$|z(\pm A, t)| = c_1 e^{c_2 A^2} \frac{1}{\sqrt{1-4c_3t}} \exp\left\{ c_3 \left(\frac{1}{1-4c_3t} - 1 \right) A^2 \right\}$$

$$\geq c_1 e^{c_2 A^2}$$

Quindi $|v(x,t)| \le z(x,t)$ su $\partial_p R$

In particolare per il princ. di confronto

$$|v(x_1,t_1)| \le z(x_1,t_1) = C_1 \exp\{(C_2 - C_3)A^2\} w(x_1,t_1)$$

e per $A \to +\infty$

$$|v(x_1,t_1)| = 0.$$

$\Longrightarrow$ $v \equiv 0$ in $\mathbb{R} \times [0,T]$

ripetendo da $t = T$ si ottiene

$$v \equiv 0 \text{ in } \mathbb{R} \times [0, 2T]$$

e per induzione

$$v \equiv 0 \text{ in } \mathbb{R} \times [0, +\infty).$$