1 Fourier transform

Definition 1.1 (Fourier transform)**.** For $f \in L^1(\mathbb{R}^d, \mathbb{C})$ we call its Fourier transform the function defined by the following formula

$$\widehat{f}(\xi) := (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{-\mathrm{i}\xi\cdot x} f(x) dx. \tag{1.1}$$

We use also the notation $\mathcal{F}f(\xi) = \widehat{f}(\xi)$.

Example 1.2. We have for any $\varepsilon > 0$

$$e^{-\varepsilon \frac{|\xi|^2}{2}} = (2\pi\varepsilon)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{-\mathrm{i}\xi\cdot x} e^{-\frac{|x|^2}{2\varepsilon}} dx. \tag{1.2}$$

We set also

$$\mathcal{F}^* f(\xi) := (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{\mathrm{i}\xi\cdot x} f(x) dx. \tag{1.3}$$

We have what follows.

Theorem 1.3. *The following facts hold.*

(1) We have $|\widehat{f}(\xi)| \le (2\pi)^{-\frac{d}{2}} \|f\|_{L^1(\mathbb{R}^d,\mathbb{C})}$*. So in particular we have*

$$\|\mathcal{F}f\|_{L^\infty(\mathbb{R}^d,\mathbb{C})} \le (2\pi)^{-\frac{d}{2}} \|f\|_{L^1(\mathbb{R}^d,\mathbb{C})}. \tag{1.4}$$

(2) (Riemann– Lebesgue Lemma) We have $\lim_{\xi\to\infty} \widehat{f}(\xi) = 0$.

(3) The bounded linear operator $\mathcal{F} : L^1(\mathbb{R}^d, \mathbb{C}) \to L^\infty(\mathbb{R}^d, \mathbb{C})$ *has values in the following space* $C_0(\mathbb{R}^d, \mathbb{C}) \subset L^\infty(\mathbb{R}^d, \mathbb{C})$

$$C_0(\mathbb{R}^d, \mathbb{C}) := \{g \in C^0(\mathbb{R}^d, \mathbb{C}) : \lim_{x\to\infty} g(x) = 0\}. \tag{1.5}$$

(4) $\mathcal{F}$ *defines an isomorphism of the space of Schwartz functions* $\mathcal{S}(\mathbb{R}^d, \mathbb{C})$ *into itself.*

(5) $\mathcal{F}$ *defines an isomorphism of the space of tempered distributions* $\mathcal{S}'(\mathbb{R}^d, \mathbb{C})$ *into itself. We have* $\mathcal{F}[\partial_{x_j} f] = -\mathrm{i}\xi_j \mathcal{F}f$.

(6) For $f, g \in L^1(\mathbb{R}^d, \mathbb{C})$ *we have*

$$\widehat{f * g}(\xi) = (2\pi)^{\frac{d}{2}} \widehat{f}(\xi)\widehat{g}(\xi).$$

□

Theorem 1.4 (Fourier transform in L^2)**.** *The following facts hold.*

(1) For a function $f \in L^1(\mathbb{R}^d,\mathbb{C}) \cap L^2(\mathbb{R}^d,\mathbb{C})$ we have that $\widehat{f} \in L^2(\mathbb{R}^d,\mathbb{C})$ and $\|\widehat{f}\|_{L^2} = \|f\|_{L^2}$. An operator

$$\mathcal{F} : L^2(\mathbb{R}^d,\mathbb{C}) \to L^2(\mathbb{R}^d,\mathbb{C}) \tag{1.6}$$

remains defined. For $f \in L^2(\mathbb{R}^d,\mathbb{C})$ for any function $\varphi \in C_c(\mathbb{R}^d,\mathbb{C})$ with $\varphi = 1$ near 0 set

$$\begin{aligned} \mathcal{F}f(\xi) &:= \lim_{\lambda\nearrow\infty} (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{-\mathrm{i}\xi\cdot x} f(x)\varphi(x/\lambda)dx \\ &= \lim_{\lambda\nearrow\infty} (2\pi)^{-\frac{d}{2}} \int_{|x|\le\lambda} e^{-\mathrm{i}\xi\cdot x} f(x)dx. \end{aligned} \tag{1.7}$$

Then (1.7) defines an isometric isomorphism inside $L^2(\mathbb{R}^d,\mathbb{C})$, so in particular we have

$$\|\mathcal{F}f\|_{L^2(\mathbb{R}^d,\mathbb{C})} = \|f\|_{L^2(\mathbb{R}^d,\mathbb{C})}. \tag{1.8}$$

(2) The inverse map is defined by

$$\begin{aligned} \mathcal{F}^*f(x) &= \lim_{\lambda\nearrow\infty} (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{\mathrm{i}\xi\cdot x} f(\xi)\varphi(\xi/\lambda)d\xi \\ &= \lim_{\lambda\nearrow\infty} (2\pi)^{\frac{d}{2}} \int_{|\xi|\le\lambda} e^{\mathrm{i}\xi\cdot x} f(\xi)d\xi. \end{aligned} \tag{1.9}$$

(3) For $f \in L^1(\mathbb{R}^d,\mathbb{C}) \cap L^2(\mathbb{R}^d,\mathbb{C})$ the two definitions (1.1) and (1.7) of $\mathcal{F}$ coincide (by dominated convergence). Similarly, for $f \in L^1(\mathbb{R}^d,\mathbb{C}) \cap L^2(\mathbb{R}^d,\mathbb{C})$ the two definitions (1.3) and (1.9) of $\mathcal{F}^*$ coincide.

The above notions extend naturally to vector fields. So we have a Fourier transform $f \to \widehat{f}$ from $(L^1(\mathbb{R}^d))^d \to (C_0(\mathbb{R}^d))^d$, from $(L^2(\mathbb{R}^d))^d \to (L^2(\mathbb{R}^d))^d$, from $(\mathcal{S}(\mathbb{R}^d))^d \to (\mathcal{S}(\mathbb{R}^d))^d$ and more generally from $(\mathcal{S}'(\mathbb{R}^d))^d \to (\mathcal{S}'(\mathbb{R}^d))^d$. Notice that all these maps excepts the 1st are isomorphisms, and all are one to one maps.

The Fourier transform extends to the spaces $L^p(\mathbb{R}^d,\mathbb{C})$ for $p \in [1,2]$.

Theorem 1.5 (Hausdorff–Young). *For $p \in [1,2]$ and $f \in L^p(\mathbb{R}^d,\mathbb{C})$ then (1.7) defines a function $\mathcal{F}f \in L^{p'}(\mathbb{R}^d,\mathbb{C})$ where $p' = \frac{p}{p-1}$ and an operator remains defined which satisfies*

$$\|\mathcal{F}f\|_{L^{p'}(\mathbb{R}^d,\mathbb{C})} \le (2\pi)^{-d\left(\frac{1}{2}-\frac{1}{p'}\right)} \|f\|_{L^p(\mathbb{R}^n,\mathbb{C})}. \tag{1.10}$$

We know already cases $p = 2$ and $p = 1$. This implies that Theorem 1.5 is a consequence of the Marcel Riesz interpolation Theorem, which we discuss now.

Theorem 1.6 (Riesz–Thorin). *Let T be a linear map from $L^{p_0}(\mathbb{R}^d) \cap L^{p_1}(\mathbb{R}^d)$ to $L^{q_0}(\mathbb{R}^d) \cap L^{q_1}(\mathbb{R}^d)$ satisfying*

$$\|Tf\|_{L^{q_j}} \le M_j \|f\|_{L^{p_j}} \text{ for } j = 0,1.$$

Then for $t \in (0,1)$ and for p_t and q_t defined by

$$\frac{1}{p_t} = \frac{1-t}{p_0} + \frac{t}{p_1} \quad , \quad \frac{1}{q_t} = \frac{1-t}{q_0} + \frac{t}{q_1}$$

we have

$$\|Tf\|_{L^{q_t}} \le (M_0)^{1-t}(M_1)^t \|f\|_{L^{p_t}} \text{ for } f \in L^{p_0}(\mathbb{R}^d) \cap L^{p_1}(\mathbb{R}^d).$$

Proof of the Hausdorff–Young's Theorem. We have $\frac{1}{p} = \frac{1-t}{2} + t$ for $t = \frac{2}{p} - 1$. Hence $1 - t = 2(1 - 1/p) = \frac{2}{p'}$ and $\frac{1}{p'} = \frac{1}{2}\frac{2}{p'} + \frac{t}{\infty}$ and

$$\|\mathcal{F}\|_{L^p \to L^{p'}} \le (2\pi)^{-\frac{d}{2}t} = (2\pi)^{-\frac{d}{2}\left(\frac{2}{p}-1\right)} = (2\pi)^{d\left(\frac{1}{p}-\frac{1}{2}\right)} = (2\pi)^{d\left(\frac{1}{2}+\frac{1}{p}-1\right)} = (2\pi)^{-d\left(\frac{1}{2}-\frac{1}{p'}\right)}.$$

Proof of Riesz–Thorin's Interpolation Theorem. First of all notice that if $f \in L^a \cap L^b$ with $a < b$ then $f \in L^c$ for any $c \in (a,b)$. To see this recall Hölder

$$\|fg\|_{L^r} \le \|f\|_{L^p} \|g\|_{L^q} \text{ for } \frac{1}{r} = \frac{1}{p} + \frac{1}{q}$$

Then, since $\frac{1}{c} = \frac{t}{a} + \frac{1-t}{b}$ for $t \in (0,1)$ from $|f| = |f|^t |f|^{1-t}$ we have

$$\|f\|_{L^c} = \||f|^t |f|^{1-t}\|_{L^c} \le \||f|^t\|_{L^{\frac{a}{t}}} \||f|^{1-t}\|_{L^{\frac{b}{1-t}}} = \|f\|_{L^a}^t \|f\|_{L^b}^{1-t}.$$

For $p_t = p_0 = p_1 = \infty$ (in fact we can repeat a similar argument for $p_t = p_0 = p_1$ any fixed value in $[1, \infty]$) we then have

$$\|Tf\|_{L^{q_t}} \le \|Tf\|_{L^{q_1}}^t \|Tf\|_{L^{q_0}}^{1-t} \le (M_0)^{1-t}(M_1)^t \|f\|_{L^\infty}.$$

So let us suppose $p_t < \infty$. Then it is enough to prove

$$|\int Tfg dx| \le (M_0)^{1-t}(M_1)^t \|f\|_{L^{p_t}} \|g\|_{L^{q_t'}} = (M_0)^{1-t}(M_1)^t$$

considering only $\|f\|_{L^{p_t}} = \|g\|_{L^{q_t'}} = 1$ for simple functions $f = \sum_{j=1}^m a_j \chi_{E_j}$ where we can take the E_j to be finite measure sets mutually disjoint. If $q_t' < \infty$ we can also reduce to simple functions $g = \sum_{k=1}^N b_k \chi_{F_k}$ where the F_j are finite measure sets mutually disjoint. The case $q_t' = \infty$ reduces to the case $p_t = \infty$ by duality. In fact, see Remark 16 p. 44 [2]

$$\|T\|_{\mathcal{L}(L^{p_t}, L^1)} = \|T^*\|_{\mathcal{L}(L^\infty, L^{p'_t})}.$$

Notice that if both $p_0 < \infty$ and $p_1 < \infty$ and since we are treating $q_0 = q_1 = 1$ then $\|T\|_{\mathcal{L}(L^{p_j}, L^1)} = \|T^*\|_{\mathcal{L}(L^\infty, L^{p'_j})} \le M_j$ and so one reduces to the case $p_t = \infty$. If, say, $p_0 = \infty$, then $\|T\|_{\mathcal{L}(L^{p_1}, L^1)} = \|T^*\|_{\mathcal{L}(L^\infty, L^{p'_1})} \le M_1$ since $p_1 < \infty$, but $\|T\|_{\mathcal{L}(L^{p_0}, L^1)} = \|T^*\|_{\mathcal{L}(L^\infty, (L^\infty)')} \le M_0$, so in other words, we don't get a Lebesgue space. However, the

issue is to bound for $f \in L^{p_0} \cap L^\infty$ a $T^* f \in L^1 \cap (L^\infty)' = L^1$ where $\|T^* f\|_{(L^\infty)'} = \|T^* f\|_{L^1}$, so that one can still apply the above argument used for $p_t = \infty$.
Let us turn to the case $p_t < \infty$ and $q_t' < \infty$. For $a_j = e^{\mathrm{i}\theta_j}|a_j|$ and $b_k = e^{\mathrm{i}\psi_k}|b_k|$ the polar representations, set

$$f_z := \sum_{j=1}^{m} |a_j|^{\frac{\alpha(z)}{\alpha(t)}} e^{\mathrm{i}\theta_j} \chi_{E_j} \text{ with } \alpha(z) := \frac{1-z}{p_0} + \frac{z}{p_1}$$
$$g_z := \sum_{k=1}^{N} |b_k|^{\frac{1-\beta(z)}{1-\beta(t)}} e^{\mathrm{i}\psi_k} \chi_{F_k} \text{ with } \beta(z) := \frac{1-z}{q_0} + \frac{z}{q_1}.$$

Notice that since we are assuming $q_t' < \infty$, then $q_t > 1$ and so $\beta(t) = \frac{1}{q_t} < 1$, so that g_z is well defined. Similarly, since $p_t < \infty$ we have $\alpha(t) = \frac{1}{p_t} > 0$, so also f_z is well defined.
We consider now the function

$$F(z) = \int T f_z g_z dx.$$

Our goal is to prove $|F(t)| \le M_0^{1-t} M_1^t$.
$F(z)$ is holomorphic in $0 < \operatorname{Re} z < 1$, continuous and bounded in $0 \le \operatorname{Re} z \le 1$. Boundedness follows from estimates like

$$||a_j|^{\frac{\alpha(z)}{\alpha(t)}}| = |a_j|^{\frac{\alpha(\operatorname{Re} z)}{\alpha(t)}} \text{ which is bounded for } 0 \le \operatorname{Re} z \le 1.$$

We have $F(t) = \int T f g dx$ since $f_t = f$ and $g_t = g$.
By the 3 lines lemma, see below, which yields $|F(z)| \le M_0^{1-\operatorname{Re} z} M_1^{\operatorname{Re} z}$, our theorem is a consequence of the following two inequalities

$$|F(z)| \le M_0 \text{ for } \operatorname{Re} z = 0 \; ;$$
$$|F(z)| \le M_1 \text{ for } \operatorname{Re} z = 1 \; .$$

For $z = \mathrm{i}y$ we have for $p_0 < \infty$

$$|f_{\mathrm{i}y}|^{p_0} = \sum_{j=1}^{m} ||a_j|^{\frac{\alpha(\mathrm{i}y)}{\alpha(t)}}|^{p_0} \chi_{E_j} = \sum_{j=1}^{m} ||a_j|^{\frac{\frac{1}{p_0} + \mathrm{i}y\left(\frac{1}{p_1} - \frac{1}{p_0}\right)}{\frac{1}{p_t}}}|^{p_0} \chi_{E_j}$$
$$= \sum_{j=1}^{m} ||a_j|^{\mathrm{i}y p_t \left(\frac{1}{p_1} - \frac{1}{p_0}\right)} |a_j|^{\frac{p_t}{p_0}}|^{p_0} \chi_{E_j} = \sum_{j=1}^{m} |a_j|^{p_t} \chi_{E_j} = |f|^{p_t}.$$

This implies

$$\|f_{\mathrm{i}y}\|_{p_0} = (\int_{\mathbb{R}^d} |f_{\mathrm{i}y}|^{p_0} dx)^{\frac{1}{p_0}} = (\int_{\mathbb{R}^d} |f|^{p_t} dx)^{\frac{1}{p_0}} = 1. \tag{1.11}$$

Notice that we have also $\|f_{\mathrm{i}y}\|_\infty = 1$ when $p_0 = \infty$.

Proceeding similarly, using $1-\beta(z)=\frac{1-z}{q_0'}+\frac{z}{q_1'}$, for $z=\mathrm{i}y$ and $q_0'<\infty$ we have

$$|g_{\mathrm{i}y}|^{q_0'}=\sum_{k=1}^{N}||b_k|^{\frac{1-\beta(\mathrm{i}y)}{1-\beta(t)}}|^{q_0'}\chi_{F_k}=\sum_{k=1}^{N}||b_k|^{\frac{\mathrm{i}y\left(\frac{1}{q_1'}-\frac{1}{q_0'}\right)}{\frac{1}{q_t'}}}|b_k|^{\frac{\frac{1}{q_0'}}{\frac{1}{q_t'}}}|^{q_0'}\chi_{F_k}=\sum_{j=1}^{N}|b_k|^{q_t'}\chi_{F_k}=|g|^{q_t'}.$$

This implies

$$\|g_{\mathrm{i}y}\|_{q_0'}=(\int_{\mathbb{R}^d}|g_{\mathrm{i}y}|^{q_0'}dx)^{\frac{1}{q_0'}}=(\int_{\mathbb{R}^d}|g|^{q_t'}dx)^{\frac{1}{q_0'}}=1. \tag{1.12}$$

Notice that we have also $\|g_{\mathrm{i}y}\|_\infty=1$ when $q_0'=\infty$..
Then

$$|F(\mathrm{i}y)|\le\|Tf_{\mathrm{i}y}\|_{q_0}\|g_{\mathrm{i}y}\|_{q_0'}\le M_0\|f_{\mathrm{i}y}\|_{p_0}\|g_{\mathrm{i}y}\|_{q_0'}=M_0.$$

By a similar argument

$$|f_{1+\mathrm{i}y}|^{p_1}=|f|^{p_t}$$
$$|g_{1+\mathrm{i}y}|^{q_1'}=|g|^{q_t'}.$$

Indeed by $\alpha(1+\mathrm{i}y)=\frac{1+\mathrm{i}y}{p_1}-\frac{\mathrm{i}y}{p_0}$

$$|f_{1+\mathrm{i}y}|^{p_1}=\sum_{j=1}^{m}||a_j|^{\frac{\alpha(1+\mathrm{i}y)}{\alpha(t)}}|^{p_1}\chi_{E_j}=\sum_{j=1}^{m}||a_j|^{\frac{\frac{1}{p_1}+\mathrm{i}y\left(\frac{1}{p_1}-\frac{1}{p_0}\right)}{\frac{1}{p_t}}}|^{p_1}\chi_{E_j}$$
$$=\sum_{j=1}^{m}||a_j|^{\frac{p_t}{p_1}}|^{p_1}\chi_{E_j}=\sum_{j=1}^{m}|a_j|^{p_t}\chi_{E_j}=|f|^{p_t}$$

and by $1-\beta(1+\mathrm{i}y)=\frac{1+\mathrm{i}y}{q_1'}-\frac{\mathrm{i}y}{q_0'}$

$$|g_{1+\mathrm{i}y}|^{q_1'}=\sum_{k=1}^{N}||b_k|^{\frac{1-\beta(1+\mathrm{i}y)}{1-\beta(t)}}|^{q_1'}\chi_{F_k}=\sum_{k=1}^{N}||b_k|^{\frac{\mathrm{i}y\left(\frac{1}{q_1'}-\frac{1}{q_0'}\right)}{\frac{1}{q_t'}}}|b_k|^{\frac{\frac{1}{q_1'}}{\frac{1}{q_t'}}}|^{q_1'}\chi_{F_k}=\sum_{j=1}^{N}|b_k|^{q_t'}\chi_{F_k}=|g|^{q_t'}.$$

Finally

$$|F(1+\mathrm{i}y)|\le\|Tf_{1+\mathrm{i}y}\|_{q_1}\|g_{1+\mathrm{i}y}\|_{q_1'}\le M_1\|f_{1+\mathrm{i}y}\|_{p_1}\|g_{1+\mathrm{i}y}\|_{q_1'}=M_1\|f\|_{p_t}^{\frac{p_t}{p_1}}\|g\|_{q_t'}^{\frac{q_t'}{q_1'}}=M_1.$$

□

Here we have used the following lemma.

Lemma 1.7 (Three Lines Lemma)**.** *Let $F(z)$ be holomorphic in the strip $0<\operatorname{Re}z<1$, continuous and bounded in $0\le\operatorname{Re}z\le1$ and such that*

$$|F(z)|\le M_0 \text{ for } \operatorname{Re}z=0\ ;$$
$$|F(z)|\le M_1 \text{ for } \operatorname{Re}z=1\ .$$

Then we have $|F(z)| \le M_0^{1-\mathrm{Re}\, z} M_1^{\mathrm{Re}\, z}$ for all $0 < \mathrm{Re}\, z < 1$.

Proof. Let us start with the special case $M_0 = M_1 = 1$ and set $B := \|F\|_{L^\infty}$. Set $h_\epsilon(z) := (1+\epsilon z)^{-1}$ with $\epsilon > 0$. Since $\mathrm{Re}(1+\epsilon z) = 1 + \epsilon x \ge 1$ it follows $|h_\epsilon(z)| \le 1$ in the strip. Furthermore $\mathrm{Im}(1+\epsilon z) = \epsilon y$ implies also $|h_\epsilon(z)| \le |\epsilon y|^{-1}$. Consider now the two horizontal lines $y = \pm B/\epsilon$ and let R be the rectangle $0 \le x \le 1$ and $|y| \le B/\epsilon$. In $|y| \ge B/\epsilon$ we have

$$|F(z)h_\epsilon(z)| \le \frac{B}{|\epsilon y|} \le \frac{B}{|\epsilon B/\epsilon|} = 1.$$

On the other hand by the maximum modulus principle

$$\sup_R |F(z)h_\epsilon(z)| = \sup_{\partial R} |F(z)h_\epsilon(z)| \le 1,$$

where on the horizontal sides the last inequality follows from the previous inequality and on the vertical sides follows from $|F(z)| \le 1$ for $\mathrm{Re}\, z = 0, 1$ and from $|h_\epsilon(z)| \le 1$.
Hence in the whole strip $0 \le x \le 1$ we have $|F(z)h_\epsilon(z)| \le 1$ for any $\epsilon > 0$. This implies

$$\lim_{\epsilon \searrow 0} |F(z)h_\epsilon(z)| = |F(z)| \le 1$$

in the whole strip $0 \le x \le 1$.
In the general case $(M_0, M_1) \ne (1,1)$ set $g(z) := M_0^{1-z} M_1^z$. Notice that

$$\begin{aligned} &g(z) = e^{(1-z)\log M_0} e^{z \log M_1} \Rightarrow |g(z)| = M_0^{1-x} M_1^x \Rightarrow \\ &\min(M_0, M_1) \le |g(z)| \le \max(M_0, M_1). \end{aligned}$$

So $F(z)g^{-1}(z)$ satisfies the hypotheses of the case $M_0 = M_1 = 1$ and so $|F(z)| \le |g(z)| = M_0^{1-\mathrm{Re}\, z} M_1^{\mathrm{Re}\, z}$

□

We consider now for $\triangle := \sum_j \frac{\partial^2}{\partial x_j^2}$ and for $f \in \mathcal{S}'(\mathbb{R}^d, \mathbb{C})$ the heat equation

$$u_t - \triangle u = 0\,, \quad u(0,x) = f(x). \tag{1.13}$$

By applying $\mathcal{F}$ we transform the above problem into

$$\widehat{u}_t + |\xi|^2 \widehat{u} = 0\,, \quad \widehat{u}(0,\xi) = \widehat{f}(\xi).$$

This yields $\widehat{u}(t,\xi) = e^{-t|\xi|^2} \widehat{f}(\xi)$. Notice that since $\widehat{f} \in \mathcal{S}'(\mathbb{R}^d, \mathbb{C})$ and $e^{-t|\cdot|^2} \in \mathcal{S}(\mathbb{R}^d, \mathbb{C})$ for any $t > 0$, the last product is well defined. Furthermore, we have $\widehat{u}(t,\cdot) \in C^0([0,+\infty), \mathcal{S}'(\mathbb{R}^d, \mathbb{C}))$ and, as a consequence, since $\mathcal{F}$ is an isomorphism of $\mathcal{S}'(\mathbb{R}^d, \mathbb{C})$ also $u(t,\cdot) \in C^0([0,+\infty), \mathcal{S}'(\mathbb{R}^n, \mathbb{C}))$.
We have $e^{-t|\xi|^2} = \widehat{G}(t,\xi)$ with $G(t,x) = (2t)^{-\frac{d}{2}} e^{-\frac{|x|^2}{4t}}$. Then, from $\widehat{u}(t,\xi) = \widehat{G}(t,\xi)\widehat{f}(\xi)$ it follows $u(t,x) = (2\pi)^{-\frac{d}{2}} G(t,\cdot) * f(x)$. In particular, for $f \in L^p(\mathbb{R}^d, \mathbb{C})$, we have

$$u(t,x) = (4\pi t)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{-\frac{|x-y|^2}{4t}} f(y)dy.$$

Notice that by (1.2) we have

$$(4\pi t)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{-\frac{|x|^2}{4t}} dx = 1.$$

We will write

$$e^{t\triangle} f(x) := (4\pi t)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{-\frac{|x-y|^2}{4t}} f(y) dy. \tag{1.14}$$

Notice that for $p \geq 1$ we have $\|e^{t\triangle} f\|_{L^p(\mathbb{R}^d)} \leq \|f\|_{L^p(\mathbb{R}^d)}$ and for $f \in L^1(\mathbb{R}^d)$ and any $x \in \mathbb{R}^d$

$$|e^{t\triangle} f(x)| \leq (4\pi t)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{-\frac{|x-y|^2}{4t}} |f(y)| dy \leq (4\pi t)^{-\frac{d}{2}} \int_{\mathbb{R}^d} |f(y)| dy = (4\pi t)^{-\frac{d}{2}} \|f\|_{L^1(\mathbb{R}^d)}.$$

We set also $K_t(x) := (4\pi t)^{-\frac{d}{2}} e^{-\frac{|x|^2}{4t}}$. Then $e^{t\triangle} f = K_t * f$. $K_t(x-y)$ is the *Heath Kernel*. As a corollary to the Riesz–Thorin Theorem we obtain the following result.

Corollary 1.8. *For any $q \geq p \geq 1$ and any $f \in L^p(\mathbb{R}^d)$ we have*

$$\|e^{t\triangle} f\|_{L^q(\mathbb{R}^d)} \leq (4\pi t)^{-\frac{d}{2}\left(\frac{1}{p}-\frac{1}{q}\right)} \|f\|_{L^p(\mathbb{R}^d)}. \tag{1.15}$$

Proof. Notice that (1.15) is true for $p = q$ and for $q = \infty$ and $p = 1$. For $q > p = 1$ Riesz–Thorin and $\dfrac{1}{q} = \dfrac{1-\frac{1}{q}}{\infty} + \dfrac{\frac{1}{q}}{1}$ yields

$$\|e^{t\triangle}\|_{L^1 \to L^q} \leq \|e^{t\triangle}\|_{L^1 \to L^\infty}^{1-\frac{1}{q}} \|e^{t\triangle}\|_{L^1 \to L^1}^{\frac{1}{q}} \leq (4\pi t)^{-\frac{d}{2}\left(1-\frac{1}{q}\right)} = (4\pi t)^{-\frac{d}{2q'}} \text{ with } q' = \frac{q}{q-1}.$$

Next, for $1 < p < q$ we have $\frac{1}{p} = \alpha + \frac{1-\alpha}{q} = \frac{1}{q} + \frac{\alpha}{q'}$ s.t. $\alpha = q'\left(\frac{1}{p} - \frac{1}{q}\right)$. Then

$$\|e^{t\triangle}\|_{L^p \to L^q} \leq \|e^{t\triangle}\|_{L^1 \to L^q}^{\alpha} \|e^{t\triangle}\|_{L^q \to L^q}^{1-\alpha} \leq (4\pi t)^{-\frac{d}{2q'}\alpha} = (4\pi t)^{-\frac{d}{2}\left(\frac{1}{p}-\frac{1}{q}\right)}.$$

□

Theorem 1.9. *$\rho \in L^1(\mathbb{R}^d)$ be s.t. $\int \rho(x) dx = 1$. Set $\rho_\epsilon(x) := \epsilon^{-d} \rho(x/\epsilon)$. Consider $C_c(\mathbb{R}^d, \mathbb{C})$ and for each $p \in [1, \infty]$ let $\overline{C_c(\mathbb{R}^d, \mathbb{C})}_p$ be the closure of $C_c(\mathbb{R}^d, \mathbb{C})$ in $L^p(\mathbb{R}^d, \mathbb{C})$, so that $\overline{C_c(\mathbb{R}^d, \mathbb{C})}_p = L^p(\mathbb{R}^d, \mathbb{C})$ for $p < \infty$ and $\overline{C_c(\mathbb{R}^d, \mathbb{C})}_\infty = C_0(\mathbb{R}^d, \mathbb{C}) \subsetneqq L^\infty(\mathbb{R}^d, \mathbb{C})$. Then for any $f \in \overline{C_c(\mathbb{R}^d, \mathbb{C})}_p$ we have*

$$\lim_{\epsilon \searrow 0} \rho_\epsilon * f = f \text{ in } L^p(\mathbb{R}^d, \mathbb{C}). \tag{1.16}$$

In particular we have

$$\lim_{t \searrow 0} e^{t\triangle} f = f \text{ in } L^p(\mathbb{R}^d, \mathbb{C}). \tag{1.17}$$

Proof. Clearly, (1.17) is a special case of (A.10) setting $\epsilon = \sqrt{t}$ and $\rho(x) = (4\pi)^{-\frac{d}{2}} e^{-\frac{|x|^2}{4}}$. To prove (A.10) we start with $f \in C_c(\mathbb{R}^d, \mathbb{C})$. In this case

$$\rho_\epsilon * f(x) - f(x) = \int_{\mathbb{R}^d} (f(x - \epsilon y) - f(x))\rho(y)dy$$

so that, by Minkowski inequality and for $\Delta(y) := \|f(\cdot - y) - f(\cdot)\|_{L^p}$, we have

$$\|\rho_\epsilon * f(x) - f(x)\|_{L^p} \le \int |\rho(y)|\Delta(\epsilon\ y)dy.$$

Now we have $\lim_{y\to 0} \Delta(y) = 0$ and $\Delta(y) \le 2\|f\|_{L^p}$. So, by dominated convergence we get

$$\lim_{\epsilon \searrow 0} \|\rho_\epsilon * f(x) - f(x)\|_{L^p} = \lim_{\epsilon \searrow 0} \int |\rho(y)|\Delta(\epsilon\ y)dy = 0.$$

So this proves (A.10) for $f \in C_c(\mathbb{R}^d, \mathbb{C})$. The general case is proved by a density argument. □

2 Some spaces of functions

We start by defining L^2 Sobolev spaces. We will introduce the *homogeneous* Sobolev spaces $\dot{H}^k(\mathbb{R}^d)$ and the standard Sobolev spaces $H^k(\mathbb{R}^d)$. For $\xi \in \mathbb{R}^d$ let $\langle\xi\rangle = \sqrt{1 + |\xi|^2}$ be the *Japanese bracket*. For a tempered distribution u we denote by $\widehat{u}$ its Fourier transform. The following spaces are formed by tempered distributions u s.t. $\widehat{u}$ is in $L^1_{loc}(\mathbb{R}^d)$ for $s \in \mathbb{R}$:

$$\dot{H}^s(\mathbb{R}^d) \text{ defined with } \|u\|_{\dot{H}^s(\mathbb{R}^d)} := \||\xi|^s\widehat{u}\|_{L^2(\mathbb{R}^d)} \ ; \tag{2.1}$$

$$H^s(\mathbb{R}^d) \text{ defined with } \|u\|_{H^s(\mathbb{R}^d)} := \|\langle\xi\rangle^s\widehat{u}\|_{L^2(\mathbb{R}^d)} \ . \tag{2.2}$$

The following lemma is elementary.

Lemma 2.1. *The following statements are true.*

- $L^2(\mathbb{R}^d) \to H^s(\mathbb{R}^d)$ *defined by* $f \to \mathcal{F}^*\left(\frac{\widehat{f}}{\langle\xi\rangle^s}\right)$ *is an isometric isomorphism and all the* $H^s(\mathbb{R}^d)$ *are Hilbert spaces with inner product* $\langle f, g\rangle_{H^s} = \langle\langle\xi\rangle^s\widehat{f}, \langle\xi\rangle^s\widehat{g}\rangle_{L^2}$.
- *We have* $\mathcal{S}(\mathbb{R}^d) \subseteq \dot{H}^s(\mathbb{R}^d)$ *if and only if* $s > -d/2$.
- *The* $\dot{H}^s(\mathbb{R}^d)$ *have an inner product defined by* $\langle f, g\rangle_{\dot{H}^s} = \langle|\xi|^s\widehat{f}, |\xi|^s\widehat{g}\rangle_{L^2}$

While the $\dot{H}^s(\mathbb{R}^d)$ have an inner product, in general they are not complete topological vector spaces.

Proposition 2.2. *For* $s < d/2$ *the space* $\dot{H}^s(\mathbb{R}^d)$ *is complete and the Fourier transform establishes an isometric isomorphism* $\mathcal{F} : \dot{H}^s(\mathbb{R}^d) \to L^2(\mathbb{R}^d, |\xi|^{2s}d\xi)$.

The above proposition is a consequence of the following lemma.

Lemma 2.3. *Let $s < \frac{d}{2}$. Then we have the following facts.*

- $L^2(\mathbb{R}^d, |\xi|^{2s} d\xi) \subset L^1_{loc}(\mathbb{R}^d, d\xi)$
- $L^2(\mathbb{R}^d, |\xi|^{2s} d\xi) \subset \mathcal{S}'(\mathbb{R}^d)$
- *The Fourier transform $\mathcal{F} : \mathcal{S}'(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$ is s.t. $\mathcal{F}\left(\dot{H}^s(\mathbb{R}^d)\right) = L^2(\mathbb{R}^d, |\xi|^{2s} d\xi)$ and establishes an isometry between these two spaces.*

Proof. Let $g \in L^2(\mathbb{R}^d, |\xi|^{2s} d\xi)$. Obviously $g \in L^1_{loc}(\mathbb{R}^d \backslash \{0\}, d\xi)$. Let now $B = \{\xi \in \mathbb{R}^d : |\xi| \le 1\}$. Then

$$\int_B |g(\xi)| d\xi \le \left(\int_B |\xi|^{2s} |g(\xi)|^2 d\xi\right)^{\frac{1}{2}} \left(\int_B |\xi|^{-2s} d\xi\right)^{\frac{1}{2}}$$

$$\le \sqrt{\mathrm{vol}(S^{d-1})} (\int_0^1 r^{d-1-2s} dr)^{\frac{1}{2}} \|g\|_{L^2(\mathbb{R}^d, |\xi|^{2s} d\xi)} = \sqrt{\frac{\mathrm{vol}(S^{d-1})}{d-2s}} \|g\|_{L^2(\mathbb{R}^d, |\xi|^{2s} d\xi)}.$$

Next, we check that $L^2(\mathbb{R}^d, |\xi|^{2s} d\xi) \subset \mathcal{S}'(\mathbb{R}^d)$. We split $g = \chi_B g + \chi_{B^c} g$. Then $\chi_B g \in L^1(\mathbb{R}^d, d\xi)$ implies $\chi_B g \in \mathcal{S}'(\mathbb{R}^d)$. On the other hand we have $\chi_{B^c} g \in L^2(\mathbb{R}^d, \langle\xi\rangle^{2s} d\xi)$. This in turn implies $\chi_{B^c} g \in \mathcal{S}'(\mathbb{R}^d)$, where the embedding $L^2(\mathbb{R}^d, \langle\xi\rangle^{2\sigma} d\xi) \subset \mathcal{S}'(\mathbb{R}^d)$ for any $\sigma \in \mathbb{R}$ follows from

$$\int_{\mathbb{R}^d} f(\xi)\varphi(\xi) d\xi = \int_{\mathbb{R}^d} \langle\xi\rangle^{\sigma} f(\xi) \langle\xi\rangle^{-\sigma} \varphi(\xi) d\xi \le \|f\|_{L^2(\mathbb{R}^d, \langle\xi\rangle^{2\sigma} d\xi)} (\int_{\mathbb{R}^d} \langle\xi\rangle^{-2\sigma} \varphi(\xi) d\xi)^{\frac{1}{2}}$$

$$\le \|f\|_{L^2(\mathbb{R}^d, \langle\xi\rangle^{2\sigma} d\xi)} (\int_{\mathbb{R}^d} \langle\xi\rangle^{-2\sigma-m} d\xi)^{\frac{1}{2}} \|\langle\xi\rangle^m \varphi\|_{L^\infty(\mathbb{R}^d)}$$

for m chosen s.t. $2\sigma + m > d$. □

Remark 2.4. For $s \ge \frac{d}{2}$ the space $\dot{H}^s(\mathbb{R}^d)$ is not a complete space for the norm indicated. In particular, the Fourier trasform defines an embedding $\dot{H}^s(\mathbb{R}^d) \overset{\mathcal{F}}{\hookrightarrow} L^2(\mathbb{R}^d, |\xi|^{2s} d\xi)$ with image which is strictly contained and dense in $L^2(\mathbb{R}^d, |\xi|^{2s} d\xi)$. The fact that the image is dense can be seen observing that $C_c^\infty(\mathbb{R}^d \backslash \{0\})$ is dense in $L^2(\mathbb{R}^d, |\xi|^{2s} d\xi)$ and we have $\mathcal{F}\dot{H}^s(\mathbb{R}^d) \supseteq C_c^\infty(\mathbb{R}^d \backslash \{0\})$.

For $s = \frac{d}{2} + \varepsilon_0$ with $\varepsilon_0 > 0$, if we pick $f \in C_c^\infty(\mathbb{R}^d)$ with $f(0) \neq 0$, then $\frac{f(\xi)}{|\xi|^{d+\frac{\varepsilon_0}{2}}}$ is a Borel function not contained in $L^1_{loc}(\mathbb{R}^d, d\xi)$. But $|\xi|^{2s} \left|\frac{f(\xi)}{|\xi|^{d+\frac{\varepsilon_0}{2}}}\right|^2 = \frac{|f(\xi)|^2}{|\xi|^{d-\varepsilon_0}} \in L^1(\mathbb{R}^d, d\xi)$ implies that $\frac{f(\xi)}{|\xi|^{d+\frac{\varepsilon_0}{2}}} \in L^2(\mathbb{R}^d, |\xi|^{2s} d\xi)$.

For $s = \frac{d}{2}$ consider $f(\xi) = \displaystyle\sum_{k=1}^{\infty} \frac{2^{kd}}{k} \chi_{[3/4,5/4]}(2^k|\xi|)$. Notice that for each ξ, at most one term of the sum is non zero, because $[2^{-k}3/4, 2^{-k}5/4] \cap [2^{-j}3/4, 2^{-j}5/4] = \emptyset$ for $j \neq k$. Indeed, if $j < k$ then

$$2^{-k}5/4 \le 2^{-(j-1)}5/4 < 2^{-j}3/4 \text{ where the latter follows from } 5 < 6.$$

Then $|\xi|^{\frac{d}{2}}|f(\xi)| \in L^2(\mathbb{R}^d, d\xi)$ since

$$\int_{\mathbb{R}^d} |\xi|^d |f(\xi)|^2 d\xi = \sum_{k=1}^{\infty} \frac{1}{k^2} 2^{2kd} \int_{\mathbb{R}^d} |\xi|^d \chi_{[3/4,5/4]}(2^k|\xi|) d\xi = \sum_{k=1}^{\infty} \frac{1}{k^2} \int_{\mathbb{R}^d} |\xi|^d \chi_{[3/4,5/4]}(|\xi|) d\xi < \infty$$

but f, which is supported in the ball $B(0, 5/4)$, is not in $L^1(\mathbb{R}^d, d\xi)$ since otherwise we would have

$$\infty > \int_{\mathbb{R}^d} |f(\xi)| d\xi \ge \sum_{k=1}^{n} \frac{1}{k} 2^{kd} \int_{\mathbb{R}^d} \chi_{[3/4,5/4]}(2^k|\xi|) d\xi = \sum_{k=1}^{n} \frac{1}{k} \int_{\mathbb{R}^d} |\xi|^d \chi_{[3/4,5/4]}(|\xi|) d\xi \overset{n\to\infty}{\to} \infty.$$

Given a vector field $u = (u^j)_{j=1}^d \in (\mathcal{S}'(\mathbb{R}^d))^d$ its divergence is

$$\mathrm{div} u = \nabla \cdot u := \sum_{j=1}^{d} \frac{\partial}{\partial x_j} u^j \, .$$

Notice that $\widehat{\mathrm{div} u} = -\mathrm{i} \sum_{j=1}^d \xi^j \widehat{u}^j$ so that a u is divergence free, that is $\mathrm{div} u = 0$, if and only if $\sum_{j=1}^d \xi^j \widehat{u}^j = 0$.

We define now an operator $\mathbb{P}$ by

$$(\mathcal{F}(\mathbb{P}u))^j = \widehat{u}^j - \frac{1}{|\xi|^2} \sum_{k=1}^{d} \xi_j \xi_k \widehat{u}^k. \tag{2.3}$$

Lemma 2.5. *Let $s < \frac{d}{2}$. Formula* (2.3) *defines a bounded operator from $(\dot{H}^{s-1}(\mathbb{R}^d))^d)$ into itself.*
$\mathbb{P}$ is a projection with image Range($\mathbb{P}$) represented by the divergence free elements of $(\dot{H}^{s-1}(\mathbb{R}^d))^d)$.
It is the orthogonal projection.
We have $\ker \mathbb{P} = \nabla \dot{H}^s(\mathbb{R}^d)$.

Proof. First of all for $\mathbb{P}$ defined by (2.3) we have

$$\|\mathbb{P}u\|_{\dot{H}^{s-1}} = \sum_{j=1}^{d} \|(\mathbb{P}u)^j\|_{\dot{H}^{s-1}} = \sum_{j=1}^{d} \||\xi|^{s-1}\mathcal{F}(\mathbb{P}u)^j\|_{L^2} = \sum_{j=1}^{d} \||\xi|^{s-1}(\widehat{u}^j - \frac{1}{|\xi|^2}\sum_{k=1}^{d} \xi_j\xi_k \widehat{u}^k)\|_{L^2}$$

$$\le \sum_{j=1}^{d} \||\xi|^{s-1}\widehat{u}^j\|_{L^2} + \sum_{j,k=1}^{d} \|\frac{\xi_j\xi_k}{|\xi|^2}\|_{L^\infty} \||\xi|^{s-1}\widehat{u}^k\|_{L^2} \le \sum_{j=1}^{d} \|u^j\|_{\dot{H}^{s-1}} + \sum_{j,k=1}^{d} \|u^k\|_{\dot{H}^{s-1}} \le (d+1)\|u\|_{\dot{H}^{s-1}}.$$

Hence this is a bounded linear operator from $(\dot{H}^{s-1}(\mathbb{R}^d))^d \to (\dot{H}^{s-1}(\mathbb{R}^d))^d$. In fact it is a projection (so $\|\mathbb{P}u\|_{\dot{H}^{s-1}} \leq \|u\|_{\dot{H}^{s-1}}$) as we will see in a moment. But first observe that

$$\mathcal{F}(\mathrm{div}\mathbb{P}u) = \mathrm{i}\sum_{j=1}^{d}\xi^j(\mathcal{F}(\mathbb{P}u))^j = \mathrm{i}\sum_{j=1}^{d}\xi^j\widehat{u}^j - \frac{\mathrm{i}}{\cancel{|\xi|^2}}\cancel{\sum_{j=1}^{d}(\xi^j)^2}\sum_{k=1}^{d}\xi_k\widehat{u}^k = 0$$

which shows that the image of $\mathbb{P}$ is formed by divergence free vector fields. Notice also that if $\mathrm{div}u = 0$, and hence $\sum_{j=1}^{d}\xi^j\widehat{u}^j = 0$, we have

$$(\mathcal{F}(\mathbb{P}u))^j = \widehat{u}^j - \frac{1}{|\xi|^2}\xi_j\underbrace{\sum_{k=1}^{d}\xi_k\widehat{u}^k}_{0} = \widehat{u}^j,$$

and so $\mathbb{P}u = u$.
Now we check that $\mathbb{P}^2 = \mathbb{P}$. We have

$$(\mathcal{F}(\mathbb{P}^2u))^j = (\mathcal{F}(\mathbb{P}u))^j - \frac{\xi_j}{|\xi|^2}\underbrace{\sum_{k=1}^{d}\xi_k(\mathcal{F}(\mathbb{P}u))^k}_{0}$$

where we use the fact checked above that $\mathrm{div}\mathbb{P}u = 0$.
All the above steps show that (2.3) defines a projection in $(\dot{H}^{s-1}(\mathbb{R}^d))^d$ whose image is formed by the divergence free operators in $(\dot{H}^{s-1}(\mathbb{R}^d))^d$.
Pick now $V \in \dot{H}^s$. Then $\nabla V \in (\dot{H}^{s-1}(\mathbb{R}^d))^d$ and we have

$$(\mathcal{F}(\mathbb{P}\nabla V))^j = -\mathrm{i}\left(\xi_j - \sum_{k=1}^{d}\frac{\xi_j\xi_k^2}{|\xi|^2}\right)\widehat{V}(\xi) = 0.$$

Hence $\ker\mathbb{P} \supseteq \nabla\dot{H}^s(\mathbb{R}^d)$. We now show $\ker\mathbb{P} \subseteq \nabla\dot{H}^s(\mathbb{R}^d)$.
If $\mathbb{P}u = 0$ then

$$\widehat{u}^j = -\mathrm{i}\xi_j\widehat{V}(\xi) \text{ where } \widehat{V}(\xi) := \frac{\mathrm{i}}{|\xi|^2}\sum_{k=1}^{d}\xi_k\widehat{u}^k$$

It is easy to see that $\widehat{V} \in L^2(\mathbb{R}^d, |\xi|^{2s}d\xi)$ and in view of the identification of this space with $\dot{H}^s(\mathbb{R}^d)$ through the Fourier transform when $s < \frac{d}{2}$ we conclude that $V \in \dot{H}^s(\mathbb{R}^d)$ with $\nabla V = u$.

□

For $u \in \dot{H}^k(\mathbb{R}^d)$ and $\lambda > 0$ let us set $\mathbf{P}_\lambda u := \mathcal{F}^*(\chi_{|\xi|\leq\lambda}\mathcal{F}u)$. Notice that this map sends $L^2(\mathbb{R}^d)$ into itself since

$$\|\mathbf{P}_\lambda u\|_{\dot{H}^k(\mathbb{R}^d)} = \||\xi|^k\chi_{|\xi|\leq\lambda}\mathcal{F}u\|_{L^2(\mathbb{R}^d)} \leq \||\xi|^k\mathcal{F}u\|_{L^2(\mathbb{R}^d)} = \|u\|_{\dot{H}^k(\mathbb{R}^d)}.$$

Notice that $\mathbf{P}_\lambda$ is a projection, that is $\mathbf{P}_\lambda^2 = \mathbf{P}_\lambda$, by

$$\mathbf{P}_\lambda^2 u = \mathbf{P}_\lambda \circ \mathbf{P}_\lambda u = \mathcal{F}^*(\chi_{|\xi|\le\lambda}\mathcal{F}\mathbf{P}_\lambda u) = \mathcal{F}^*(\chi_{|\xi|\le\lambda}^2\mathcal{F}u) = \mathcal{F}^*(\chi_{|\xi|\le\lambda}\mathcal{F}u) = \mathbf{P}_\lambda u.$$

If $\mathrm{div}u = 0$ then also $\mathrm{div}\mathbf{P}_\lambda u = 0$. Indeed

$$(\mathrm{div}u = 0 \Leftrightarrow \sum_{j=1}^d \xi^j \ \widehat{u}^j = 0) \Rightarrow \mathcal{F}(\mathrm{div}\mathbf{P}_\lambda u) = \sum_{j=1}^d \xi^j \chi_{|\xi|\le\lambda}\widehat{u}^j = \chi_{|\xi|\le\lambda}\sum_{j=1}^d \xi^j \ \widehat{u}^j = 0,$$

which in turn implies $\mathrm{div}\mathbf{P}_\lambda u = 0$.

3 Hardy Littlewood maximal function and Sobolev estimates

Let $f \in L^1_{loc}(\mathbb{R}^d)$ and consider (for $B(x,r)$ the ball of center x and radius r in $\mathbb{R}^d$) averages

$$A_r f(x) = \frac{1}{\mathrm{vol}(B(x,r))}\int_{B(x,r)} f(y)dy.$$

Notice that for any $r > 0$ the function $x \to A_r f(x)$ is continuous. Indeed, fix $\delta_0 > 0$ and consider $\delta x \in B(0,\delta_0)$. Then by the triangular inequality $B(x+\delta x, r) \subset B(x, r+\delta_0)$. So, for $\delta x \in B(0,\delta_0)$

$$A_r f(x) - A_r f(x+\delta x) = \frac{1}{\mathrm{vol}(B(0,1))r^d}\int_{B(x,r+\delta_0)} \left(\chi_{B(x,r)\backslash B(x+\delta x,r)}(y) - \chi_{B(x+\delta x,r)\backslash B(x,r)}(y)\right) f(y)dy$$

with for any y

$$\left(\chi_{B(x,r)\backslash B(x+\delta x,r)}(y) - \chi_{B(x+\delta x,r)\backslash B(x,r)}(y)\right) \chi_{B(x,r+\delta_0)}(y) f(y) \stackrel{|\delta x|\to 0}{\to} 0.$$

By dominated convergence $A_r f(x) - A_r f(x+\delta x) \to 0$. We define

$$Mf(x) = \sup_{r>0} A_r|f|(x). \tag{3.1}$$

From the definition we conclude that Mf is lower semi continuous that is $\{x : Mf(x) > a\}$ is open for any a. It also obvious that M is sub additive:

$$M(f+g)(x) \le Mf(x) + Mg(x).$$

We have the following obvious estimate

$$|Mf(x)| \le |f|_{L^\infty(\mathbb{R}^d)}. \tag{3.2}$$

One important fact is that it is not true that M maps $L^1(\mathbb{R}^d)$ into itself. Indeed if say $K \subset \mathbb{R}^d$ is any compact set and if $B(0, c_0) \supset K$, then since for $|x| > c_0$ we have $B(x, 2|x|) \supset B(0, |x|) \supset K$, we have computing at $r = 2|x|$

$$M\chi_K(x) = \sup_{r>0} \frac{\mathrm{vol}(B(x,r) \cap K)}{\mathrm{vol}(B(0,1))r^d} \geq \frac{\mathrm{vol}(K)}{\mathrm{vol}(B(0,1))2^d|x|^d}$$

which shows that $M\chi_K \notin L^1(\mathbb{R}^d)$.
Notice that each $g \in L^1(\mathbb{R}^d)$ satisfies Chebyshev's inequality:

$$\mathrm{vol}(\{x : |g(x)| > \alpha\}) \leq \frac{|g|_{L^1(\mathbb{R}^d)}}{\alpha} \text{ for any } \alpha > 0 \tag{3.3}$$

Indeed (3.3) follows immediately from.

$$|g|_{L^1(\mathbb{R}^d)} = \int_{\mathbb{R}^d} |g(y)|dy \geq \int_{\{x:|g(x)|>\alpha\}} |g(y)|dy \geq \int_{\{x:|g(x)|>\alpha\}} \alpha dy = \alpha \mathrm{vol}(\{x : |g(x)| > \alpha\})$$

If $T : L^1(\mathbb{R}^d) \to L^1(\mathbb{R}^d)$ satisfies $\|Tf\|_{L^1(\mathbb{R}^d)} \leq A\|f\|_{L^1(\mathbb{R}^d)}$ for all $f \in L^1(\mathbb{R}^d)$ and for a fixed constant A, from (3.3) it is easy to conclude that

$$\mathrm{vol}(\{x : |Tf(x)| > \alpha\}) \leq \frac{A}{\alpha}|f|_{L^1(\mathbb{R}^d)} \text{ for any } \alpha > 0 \text{ and any } f \in L^1(\mathbb{R}^d).$$

Unfortunately we have seen that M does not map $L^1(\mathbb{R}^d)$ into itself. However we will show that it satisfies the last property. Indeed we will prove now that M *is weak* $(1,1)$ *bounded*, that is there exists a constant $A > 0$ (in fact we will prove $A = 3^d$) s.t.

$$\mathrm{vol}(\{x : Mf(x) > \alpha\}) \leq \frac{A}{\alpha}|f|_{L^1(\mathbb{R}^d)} \text{ for any } \alpha > 0 \ . \tag{3.4}$$

To prove this we consider the set $\{x : Mf(x) > \alpha\}$. Then, for any x in this set, there is a ball with center in x, which we denote by B_x, with $\int_{B_x} |f| > \alpha \mathrm{vol}(B_x)$. Pick any compact subset K of the above set, and cover it with such balls B_x. Extract now a finite cover, corresponding to finitely many points x_1, ...x_N. We have the following covering result, which we state without proof.

Theorem 3.1 (Vitali's lemma)**.** *Let B_{x_1},...,B_{x_N} be a finite number of balls in $\mathbb{R}^d$. There exists a subset of balls*

$$\{B_1, ..., B_m\} \subseteq \{B_{x_1}, ..., B_{x_N}\} \tag{3.5}$$

with the B_1...B_m pairwise disjoint, s.t.

$$vol(B_{x_1} \cup \cdots \cup B_{x_N}) \leq 3^d \sum_{j=1}^{m} vol(B_j). \tag{3.6}$$

We consider balls $B_1...B_m$ as in (3.5) and from

$$K \subset B_{x_1} \cup \cdots \cup B_{x_N} \Rightarrow \text{vol}(K) < \text{vol}(B_{x_1} \cup \cdots \cup B_{x_N}),$$

from (3.6) and from the definition of the B_{x_j} we get

$$3^{-d}\text{vol}(K) \le \sum_{j=1}^{m} \text{vol}(B_j) < \sum_{j=1}^{m} \frac{1}{\alpha} \int_{B_j} |f| \le \frac{|f|_1}{\alpha}. \tag{3.7}$$

(3.7) implies $\text{vol}(K) \le 3^d \alpha^{-1} |f|_1$. By $\text{vol}(\{x : |Mf(x)| > \alpha\}) = \sup_{K \subset \{x:|Mf(x)|>\alpha\}} \text{vol}(K)$ for compact sets K, then (3.7) implies (3.4).

(3.2) and (3.4) imply by the Marcinkiewicz Interpolation Theorem 3.2, proved below,

$$\|Mf\|_{L^p(\mathbb{R}^d)} < A_p \|f\|_{L^p(\mathbb{R}^d)} \text{ for all } p \in (1, \infty] \ . \tag{3.8}$$

We will use this result in the proof of the Hardy-Littlewood-Sobolev Theorem, and of Sobolev's estimates.

Before introducing the Marcinkiewicz interpolation Theorem, we recall that for a measurable function $g : \mathbb{R}^d \to \mathbb{R}$ the distribution function is

$$\lambda(\alpha) := \text{vol}(\{x \in \mathbb{R}^d : |g(x)| > \alpha\}).$$

Notice that $\lambda : [0, \infty) \to [0, \infty]$ is decreasing. This implies that it is measurable.
For a function $g \in L^p(\mathbb{R}^d)$ with $1 \le p < \infty$ we have

$$\begin{aligned} &\int_{\mathbb{R}^d} |g(x)|^p dx = \int_{\mathbb{R}^d} dx \int_0^{|g(x)|} p\alpha^{p-1} d\alpha = \int_0^{\infty} d\alpha p\alpha^{p-1} \int_{\{x \in \mathbb{R}^d : |g(x)| > \alpha\}} dx \\ &= \int_0^{\infty} p\alpha^{p-1} \lambda(\alpha) d\alpha \end{aligned} \tag{3.9}$$

where the 1st equality is elementary, the last follows immediately by the definition of $\lambda(\alpha)$, and the 2nd follows from Tonelli's Theorem applied to the positive measurable function $F(x, \alpha) := |\alpha|^{p-1} \chi_{\mathbb{R}_+}(|g(x)| - \alpha) \chi_{\mathbb{R}_+}(\alpha)$.

Theorem 3.2 (Marcinkiewicz Interpolation)**.** *Let $T : L^1(\mathbb{R}^d) + L^\infty(\mathbb{R}^d) \to L^1_{loc}(\mathbb{R}^d)$ be a sublinear operator s.t. for two constants A_1 and A_∞ and for all f*

$$\|Tf\|_{L^\infty(\mathbb{R}^d)} \le A_\infty \|f\|_{L^\infty(\mathbb{R}^d)} \tag{3.10}$$

$$|\{x : |Tf(x)| > \alpha\}| \le \frac{A_1}{\alpha} |f|_{L^1(\mathbb{R}^d)} \text{ for any } \alpha > 0 \ . \tag{3.11}$$

Then for any $p \in (1, \infty)$ there is a constant A_p such that for any $f \in L^p(\mathbb{R}^d)$ we have

$$\|Tf\|_{L^p(\mathbb{R}^d)} \le A_p \|f\|_{L^p(\mathbb{R}^d)}. \tag{3.12}$$

Proof. Dividing T by a constant, we can assume $A_\infty = 1$. Fix $p \in (1, \infty)$ and $f \in L^p(\mathbb{R}^d)$. For $\alpha > 0$ arbitrary set

$$f_1(x) = \begin{cases} f(x) \text{ if } |f(x)| \geq \frac{\alpha}{2} \\ 0 \text{ otherwise.} \end{cases}$$

Notice that $f_1 \in L^1(\mathbb{R}^d)$ by

$$\int_{\mathbb{R}^d} |f_1(x)| dx = \int_{\{x:|f(x)|\geq\frac{\alpha}{2}\}} |f(x)| dx \leq \frac{2^{p-1}}{\alpha^{p-1}} \int_{\mathbb{R}^d} |f(x)|^p dx.$$

Using (3.10), we get $|Tf(x)| \leq |Tf_1(x)| + \frac{\alpha}{2}$, since $\|f - f_1\|_{L^\infty(\mathbb{R}^d)} \leq \frac{\alpha}{2}$. Then

$$\{x : |Tf(x)| > \alpha\} \subseteq \{x : |Tf_1(x)| > \frac{\alpha}{2}\}.$$

We have, using (3.11),

$$\text{vol}(\{x : |Tf_1(x)| > \frac{\alpha}{2}\}) \leq A_1 \frac{2}{\alpha} \int_{\mathbb{R}^d} |f_1(x)| dx = A_1 \frac{2}{\alpha} \int_{\{x:|f(x)|\geq\frac{\alpha}{2}\}} |f(x)| dx.$$

Substituting $g = Tf$ in (3.9)

$$\begin{aligned} \int_{\mathbb{R}^d} |Tf(x)|^p dx &= \int_0^\infty p\alpha^{p-1} \text{vol}(\{x : |Tf(x)| > \alpha\}) d\alpha \\ &\leq \int_0^\infty p\alpha^{p-1} \text{vol}(\{x : |Tf_1(x)| > \frac{\alpha}{2}\}) d\alpha \leq 2A_1 \int_0^\infty p\alpha^{p-2} \int_{\{x:|f(x)|\geq\frac{\alpha}{2}\}} |f(x)| dx \\ &= 2pA_1 \int_{\mathbb{R}^d} dx |f(x)| \underbrace{\int_0^{2|f(x)|} \alpha^{p-2} d\alpha}_{\frac{2^{p-1}|f(x)|^{p-1}}{p-1}} = \frac{2^p p}{p-1} A_1 \int_{\mathbb{R}^d} |f(x)|^p dx. \end{aligned}$$

□

We will use the properties of the Hardy Littlewood Maximal function, and specifically the definition and (3.8), to prove the following important theorem.

Theorem 3.3 (Hardy-Littlewood-Sobolev inequality)**.** *For any*

$$\gamma \in (0, d) \text{ and } 1 < p < q < \infty \text{ with } \frac{1}{p} = \frac{1}{q} + \frac{d - \gamma}{d} \tag{3.13}$$

there exists a constant C s.t.

$$\| \int_{\mathbb{R}^d} f(x - y)|y|^{-\gamma} dy \|_{L^q(\mathbb{R}^d)} \leq C \|f\|_{L^p(\mathbb{R}^d)}. \tag{3.14}$$

Proof. For an $R > 0$ to be chosen momentarily, we split

$$\int_{\mathbb{R}^d} f(x-y)|y|^{-\gamma}dy = \int_{|y|<R} f(x-y)|y|^{-\gamma}dy + \int_{|y|>R} f(x-y)|y|^{-\gamma}dy.$$

We claim that

$$|\int_{|y|<R} f(x-y)|y|^{-\gamma}dy| \leq Mf(x)\int_{|y|<R} |y|^{-\gamma}dy = cR^{d-\gamma}Mf(x). \tag{3.15}$$

We assume for a moment this claim and complete the rest of the proof. By Hölder we have

$$|\int_{|y|>R} f(x-y)|y|^{-\gamma}dy| \leq \|f\|_{L^p(\mathbb{R}^d)}\||y|^{-\gamma}\chi_{\{y:|y|>R\}}\|_{L^{p'}(\mathbb{R}^d)}.$$

We have $|y|^{-\gamma}\chi_{\{y:|y|>R\}} \in L^{p'}(\mathbb{R}^d)$ exactly if $\gamma p' > d$. The latter inequality is true because

$$\frac{1}{p'} - \frac{\gamma}{d} = -\frac{1}{q} < 0 \Rightarrow \gamma p' - d = \frac{dp'}{q} > 0.$$

In this case

$$\||y|^{-\gamma}\chi_{\{y:|y|>R\}}\|_{L^{p'}(\mathbb{R}^d)} = \left(\mathrm{vol}(\mathbb{S}^{d-1})\int_{r>R} r^{-\gamma p'+d-1}dr\right)^{\frac{1}{p'}} = cR^{\frac{d}{p'}-\gamma} = cR^{-\frac{d}{q}}.$$

Hence

$$|\int_{\mathbb{R}^d} f(x-y)|y|^{-\gamma}dy| \lesssim R^{d-\gamma}Mf(x) + \|f\|_{L^p(\mathbb{R}^d)}R^{-\frac{d}{q}}.$$

Now we choose R so that the two terms on the r.h.s. are equal:

$$\frac{Mf(x)}{\|f\|_{L^p}} = R^{\gamma-d-\frac{d}{q}} = R^{-\frac{d}{p}}.$$

Then we get

$$|\int_{\mathbb{R}^d} f(x-y)|y|^{-\gamma}dy| \lesssim R^{d-\gamma}Mf(x) + \|f\|_{L^p(\mathbb{R}^d)}R^{-\frac{d}{q}} = 2\|f\|_{L^p(\mathbb{R}^d)}\left(\frac{Mf(x)}{\|f\|_{L^p}}\right)^{\frac{d}{q}\cdot\frac{p}{d}}$$
$$= 2\,(Mf(x))^{\frac{p}{q}}\,\|f\|_{L^p}^{1-\frac{p}{q}}.$$

Then

$$\|\int_{\mathbb{R}^d} f(x-y)|y|^{-\gamma}dy\|_{L^q(\mathbb{R}^d)} \lesssim \|f\|_{L^p}^{1-\frac{p}{q}}\|(Mf)^{\frac{p}{q}}\|_{L^q} = \|f\|_{L^p}^{1-\frac{p}{q}}\|(Mf)\|_{L^p}^{\frac{p}{q}} \lesssim \|f\|_{L^p}.$$

To complete the proof we need the inequality in (3.15). More generally, we prove that if $\Phi \in L^1(\mathbb{R}^d)$ is radial, positive and decreasing, then

$$|\int_{\mathbb{R}^d} f(x-y)\Phi(y)dy| \leq Mf(x)\int_{\mathbb{R}^d} \Phi(y)dy. \tag{3.16}$$

Then (3.15) is just (3.16) for $\Phi(y) = |y|^{-\gamma}\chi_{\{y:|y|<R\}}$.
Notice that (3.16) is true for radial functions of the form

$$\Phi = \sum_j a_j \chi_{B_j}$$

for $a_j > 0$, B_j a ball of center 0. Indeed

$$\sum_j a_j \int_{B_j} |f(x-y)|dy = \sum_j a_j \frac{\mathrm{vol}(B_j)}{\mathrm{vol}(B_j)} \int_{B_j} |f(x-y)|dy \le \sum_j a_j \mathrm{vol}(B_j) Mf(x) = Mf(x) \int \Phi dy.$$

In the general case the result follows from the fact that Φ can be approximated by these functions.

□

For the above proof see [14] p.354, while for the next one see [13] p.73.

Exercise 3.4. Check that, for γ, p and q as above, the operator

$$T_t f(x) = \int_{|y|\ge t} f(x-y)|y|^{-\gamma}dy$$

satisfies $T_t f \overset{t\to+\infty}{\to} 0$ in $L^q(\mathbb{R}^d)$ for any $f \in L^p(\mathbb{R}^d)$ but that it is not true that $T_t \overset{t\to+\infty}{\to} 0$ in the Banach space of linear bounded operators from $L^p(\mathbb{R}^d)$ to $L^q(\mathbb{R}^d)$ (that is, there is *strong* convergence but not *uniform* convergence to the 0 operator).

Lemma 3.5. *For any $\gamma \in (0, d)$ there exists $c_\gamma > 0$ s.t.*

$$\mathcal{F}(|\cdot|^{-\gamma})(\xi) = c_\gamma |\xi|^{\gamma-d}. \tag{3.17}$$

Proof. It is enough to show that for any $\phi \in \mathcal{S}(\mathbb{R}^d)$ we have

$$\int_{\mathbb{R}^d} |x|^{-\gamma}\phi(x)dx = c_\gamma \int_{\mathbb{R}^d} |\xi|^{\gamma-d}\widehat{\phi}(\xi)d\xi. \tag{3.18}$$

Starting from (1.2) and Plancherel we have

$$\int_{\mathbb{R}^n} \varepsilon^{-\frac{d}{2}} e^{-\frac{|x|^2}{2\varepsilon}} \phi(x)dx = \int_{\mathbb{R}^d} e^{-\varepsilon\frac{|\xi|^2}{2}} \widehat{\phi}(\xi)d\xi.$$

Now we apply to both sides $\int_0^\infty \frac{d\epsilon}{\epsilon}\epsilon^{\frac{d-\gamma}{2}}$ and commuting order of integration we obtain

$$\int_{\mathbb{R}^d} dx\phi(x) \underbrace{\int_0^\infty \varepsilon^{-\frac{\gamma}{2}} e^{-\frac{|x|^2}{2\varepsilon}} \frac{d\epsilon}{\epsilon}}_{a_\gamma |x|^{-\gamma}} = \int_{\mathbb{R}^d} d\xi \widehat{\phi}(\xi) \underbrace{\int_0^\infty \epsilon^{\frac{d-\gamma}{2}} e^{-\varepsilon\frac{|\xi|^2}{2}} \frac{d\epsilon}{\epsilon}}_{b_\gamma |\xi|^{\gamma-d}}$$

for appropriate constants a_γ and b_γ.

□

Theorem 3.6 (Sobolev Embedding Theorem)**.** *For any* $0 < s < \frac{d}{2}$ *there exists a* C *s.t. for* $\frac{1}{q} = \frac{1}{2} - \frac{s}{d}$ *we have*

$$\|f\|_{L^q(\mathbb{R}^d)} \leq C\|f\|_{\dot{H}^s(\mathbb{R}^d)}. \tag{3.19}$$

Proof. For $f \in \mathcal{S}(\mathbb{R}^d)$ we have for some fixed c

$$f(x) := (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{\mathrm{i}\xi\cdot x}|\xi|^{-s}\left(|\xi|^s\widehat{f}(\xi)\right)d\xi = c\int_{\mathbb{R}^d}|x-y|^{s-d}g(y)dy \text{ where } \widehat{g}(\xi) = |\xi|^s\widehat{f}(\xi)$$

where we used $\widehat{\varphi * T} = (2\pi)^{\frac{d}{2}}\widehat{\varphi}\,\widehat{T}$ which holds for $\varphi \in \mathcal{S}(\mathbb{R}^d)$ and $T \in \mathcal{S}'(\mathbb{R}^d)$.
Since $g \in L^2(\mathbb{R}^d)$, by the Hardy-Littlewood-Sobolev Theorem we have that $f \in L^q(\mathbb{R}^d)$ for

$$\frac{1}{q} = \frac{1}{2} - \frac{d-(d-s)}{d} = \frac{1}{2} - \frac{s}{d}$$

This extends to all $f \in \dot{H}^s(\mathbb{R}^d)$ by the density of $\mathcal{S}(\mathbb{R}^d)$ in $\dot{H}^s(\mathbb{R}^d)$ for $0 < s < \frac{d}{2}$.

□

4 Assorted inequalities

Lemma 4.1 (Interpolation of Sobolev norms)**.** *For any* $s \in [0,1]$ *and any* $k = sk_1+(1-s)k_2$ *we have*

$$\|f\|_{\dot{H}^k(\mathbb{R}^d)} \leq \|f\|^s_{\dot{H}^{k_1}(\mathbb{R}^d)}\|f\|^{1-s}_{\dot{H}^{k_2}(\mathbb{R}^d)} \textit{ for any } f \in \dot{H}^{k_1}(\mathbb{R}^d) \cap \dot{H}^{k_2}(\mathbb{R}^d). \tag{4.1}$$

In particular, for $s \in [0,1]$ *and any* $f \in H^1(\mathbb{R}^d)$

$$\|f\|_{\dot{H}^s(\mathbb{R}^d)} \leq \|f\|^{1-s}_{L^2(\mathbb{R}^d)}\|f\|^s_{\dot{H}^1(\mathbb{R}^d)} \tag{4.2}$$

Proof. (4.2) follows from (4.1) for $k_1 = 1$ and $k_2 = 0$. So let us turn to (4.1).
Obviously there is nothing to prove for $s = 0, 1$, so we can assume $s \in (0,1)$. Notice that for $p = \frac{1}{s}$ we have $p' := \frac{p}{p-1} = \frac{1}{1-s}$. Now, we have

$$\begin{aligned}
&\|f\|^2_{\dot{H}^k(\mathbb{R}^d)} = \int \left(|\xi|^{2sk_1}|\widehat{f}(\xi)|^{2s}\right)\left(|\xi|^{2(1-s)k_2}|\widehat{f}(\xi)|^{2(1-s)}\right)d\xi \\
&\leq \||\xi|^{2sk_1}|\widehat{f}(\xi)|^{2s}\|_{L^{\frac{1}{s}}(\mathbb{R}^d)}\||\xi|^{2(1-s)k_2}|\widehat{f}(\xi)|^{2(1-s)}\|_{L^{\frac{1}{1-s}}(\mathbb{R}^d)} \\
&= \||\xi|^{k_1}|\widehat{f}(\xi)|\|^{2s}_{L^2(\mathbb{R}^d)}\||\xi|^{k_1}\widehat{f}(\xi)\|^{2(1-s)}_{L^2(\mathbb{R}^d)} = \|f\|^{2s}_{\dot{H}^{k_1}(\mathbb{R}^d)}\|f\|^{2(1-s)}_{\dot{H}^{k_2}(\mathbb{R}^d)}.
\end{aligned}$$

□

Theorem 4.2 (Gagliardo–Nirenberg)**.** *If* $p \in [2,\infty)$ *is s.t.* $\frac{1}{p} > \frac{1}{2} - \frac{1}{d}$ *then there exists* C *s.t.*

$$\|f\|_{L^p(\mathbb{R}^d)} \leq C\|f\|^{1-s}_{L^2(\mathbb{R}^d)}\|f\|^s_{\dot{H}^1(\mathbb{R}^d)} \textit{ where } s = d\left(\frac{1}{2} - \frac{1}{p}\right). \tag{4.3}$$

Proof. By Sobolev, for $\frac{1}{p} = \frac{1}{2} - \frac{s}{d}$ we have

$$\|f\|_{L^p(\mathbb{R}^d)} \leq C\|f\|_{\dot{H}^s(\mathbb{R}^d)}.$$

Here s is like in the statement. Also $s = d\left(\frac{1}{2} - \frac{1}{p}\right) < 1 \Leftrightarrow \frac{1}{2} - \frac{1}{p} < \frac{1}{d}$. Finally, apply (4.2). □

Remark 4.3. For $p = 4$ and $d = 2, 3$ we have $s = d/4$ and $\|f\|_{L^4(\mathbb{R}^d)} \leq C\|f\|_{L^2(\mathbb{R}^d)}^{1-d/4}\|f\|_{\dot{H}^1(\mathbb{R}^d)}^{d/4}$.

Lemma 4.4 (Gronwall's inequality)**.** *Let $T > 0$, λ and φ two functions in $L^1(0, T)$, both ≥ 0 a.e., and C_1, C_2 two non negative constants. Let $\lambda\varphi \in L^1(0, T)$ and let*

$$\varphi(t) \leq C_1 + C_2 \int_0^t \lambda(s)\ \varphi(s)ds \text{ for a.e. } t \in (0, T).$$

Then we have

$$\varphi(t) \leq C_1 e^{C_2 \int_0^t \lambda(s)ds} \text{ for a.e. } t \in (0, T).$$

Proof. Set

$$\psi(t) := C_1 + C_2 \int_0^t \lambda(s)\ \varphi(s)ds.$$

Then $\psi(t)$ is absolutely continuous and so it is differentiable almost everywhere and we have

$$\psi'(t) = C_2\lambda(t)\ \varphi(t) \leq C_2\lambda(t)\ \psi(t) \text{ for a.e. } t \in (0, T).$$

Also, the function $\psi(t)e^{-C_2 \int_0^t \lambda(s)ds}$ is absolutely continuous with

$$\frac{d}{dt}\left(\psi(t)e^{-C_2 \int_0^t \lambda(s)ds}\right) \leq 0 \text{ for a.e. } t \in (0, T).$$

Then we have

$$\psi(t) \leq e^{C_2 \int_0^t \lambda(s)ds}\psi(0) = C_1 e^{C_2 \int_0^t \lambda(s)ds} \text{ for all } t \in (0, T).$$

Since $\varphi(t) \leq \psi(t)$ a.e., the result follows. □

5 Linear heat equation

For Sections 5–6 see [5].

Let $T \in \mathbb{R}_+$ and $f : [0, T] \to (\dot{H}^{s-1}(\mathbb{R}^d))^d$, for $d = 2, 3$, be an external force s.t. $f = \mathbb{P}f$ and consider the following heat equation:

$$\begin{cases} u_t - \nu\triangle u = f & \\ \nabla \cdot u = 0 & (t, x) \in [0, T] \times \mathbb{R}^d \\ u(0) = u_0 \in \mathbb{P}(\dot{H}^s(\mathbb{R}^d))^d & \end{cases} \tag{5.1}$$

Definition 5.1. For a fixed $s \in [0,1)$ let $f \in L^2([0,T], (\dot{H}^{s-1}(\mathbb{R}^d))^d)$ with $f = \mathbb{P}f$. Then u is a solution of (5.1) if

$$u \in L^\infty([0,T], (\dot{H}^s(\mathbb{R}^d))^d) \ , \ \nabla u \in L^2([0,T], (\dot{H}^s(\mathbb{R}^d))^{d^2}), \tag{5.2}$$

if

$$u \text{ is weakly continuous from } [0,T] \text{ into } (\dot{H}^s(\mathbb{R}^d))^d \tag{5.3}$$

(that is, if for any $\psi \in (\dot{H}^{-s}(\mathbb{R}^d))^d$ the function $t \to \langle u(t), \psi\rangle$, which is a well defined function in $L^\infty([0,T], \mathbb{R})$, is in fact in $C^0([0,T], \mathbb{R})$)
and if for any $\Psi \in C_c^\infty([0,T] \times \mathbb{R}^d, \mathbb{R}^d)$ we have

$$\langle u(t), \Psi(t)\rangle_{L^2} = \int_0^t \big(\nu\langle u(t'), \triangle\Psi(t')\rangle_{L^2} + \langle u(t'), \partial_t\Psi(t')\rangle_{L^2} + \langle f(t'), \Psi(t')\rangle_{L^2}\big)\, dt' + \langle u_0, \Psi(0)\rangle_{L^2}. \tag{5.4}$$

The following theorem yields existence, uniqueness and energy estimate for (5.1).

Theorem 5.2. *Problem* (5.1) *admits exactly one solution in the sense of the above definition. For any t the following energy estimate is satisfied:*

$$\|u(t)\|^2_{\dot{H}^s(\mathbb{R}^d)} + 2\nu \int_0^t \|\nabla u(t')\|^2_{\dot{H}^s(\mathbb{R}^d)} dt' = \|u_0\|^2_{\dot{H}^s(\mathbb{R}^d)} + 2\int_0^t \langle f(t'), u(t')\rangle_{\dot{H}^s(\mathbb{R}^d)} dt'. \tag{5.5}$$

Furthermore we have

$$u \in C^0([0,T], (\dot{H}^s(\mathbb{R}^d))^d)) \tag{5.6}$$

and the formula

$$\widehat{u}(t,\xi) = e^{-t\nu|\xi|^2}\widehat{u}_0(\xi) + \int_0^t e^{-(t-t')\nu|\xi|^2}\widehat{f}(t',\xi)dt'. \tag{5.7}$$

Proof. (**Uniqueness**). It is enough to show that the only solution of the case $u_0 = 0$ and $f = 0$ is $u = 0$. Let u be such a solution. Then

$$\langle u(t), \Psi(t)\rangle_{L^2} = \int_0^t \big(\nu\langle u(t'), \triangle\Psi(t')\rangle_{L^2} + \langle u(t'), \partial_t\Psi(t')\rangle_{L^2}\big)\, dt'.$$

Let $\Psi(t,x) = \psi(x)$ with $\psi \in C_c^\infty(\mathbb{R}^d, \mathbb{R})$. Then the above equality reduces to

$$\langle u(t), \psi\rangle_{L^2} = \nu \int_0^t \langle u(t'), \triangle\psi\rangle_{L^2},$$

which extends by density to all $\psi \in \dot{H}^{-s}(\mathbb{R}^d, \mathbb{R})$. In particular we can replace ψ by $\mathbf{P}_n\psi$ and get

$$\begin{aligned}\langle \mathbf{P}_n u(t), \psi\rangle_{L^2} &= \int_0^t \nu\langle u(t'), \triangle\mathbf{P}_n\psi\rangle_{L^2} \le \nu\|\triangle\mathbf{P}_n\psi\|_{\dot{H}^{-s}} \int_0^t \|\mathbf{P}_n u(t')\|_{\dot{H}^s} dt' \\ &\le \nu n^2 \|\psi\|_{\dot{H}^{-s}} \int_0^t \|\mathbf{P}_n u(t')\|_{\dot{H}^s} dt'\end{aligned}$$

where the integral $\int_0^t \|\mathbf{P}_n u(t')\|_{\dot{H}^s} dt'$ is well defined by $\mathbf{P}_n u \in L^\infty([0,T],(\dot{H}^s(\mathbb{R}^d))^d)$.
From the above formula

$$\|\mathbf{P}_n u(t)\|_{\dot{H}^s} \leq \nu n^2 \int_0^t \|\mathbf{P}_n u(t')\|_{\dot{H}^s} dt'$$

and hence $\|\mathbf{P}_n u(t)\|_{\dot{H}^s} = 0$ by the Gronwall inequality. This implies $u(t) = 0$ for $t \in [0,T]$.

(**Existence**). First of all, there exists a sequence (f_n) in $C^0([0,T],(\dot{H}^{s-1}(\mathbb{R}^d))^d)$ s.t. $f_n \overset{n\to\infty}{\to} f$ in $L^2([0,T],(\dot{H}^{s-1}(\mathbb{R}^d))^d)$. This follows from the density of $C_c^\infty(I,X)$ in $L^p(I,X)$ for $p < \infty$ for I an interval and X a Banach space, see Appendix A.
Applying $\mathbf{P}_n$ to (5.1) and replacing f by f_n we obtain the equation

$$\begin{cases} (u_n)_t - \nu \mathbf{P}_n \triangle u_n = \mathbf{P}_n f_n \\ u_n(0) = \mathbf{P}_n u_0 \end{cases} \tag{5.8}$$

Notice that $\mathbf{P}_n f_n \in C^0([0,T],(\dot{H}^s(\mathbb{R}^d))^d)$. Since (5.8) is a standard linear equation it admits a solution $u_n \in C^1([0,T],(\dot{H}^s(\mathbb{R}^d))^d)$. Notice furthermore that $u_n = \mathbf{P}_n u_n$ and so in particular $u_n \in C^0([0,T],(\dot{H}^r(\mathbb{R}^d))^d)$ for all $r \geq s$.
Furthermore, applying $\langle \cdot, u_n\rangle_{\dot{H}^s}$ to (5.8) and using

$$\langle \mathbf{P}_n \triangle u_n, u_n\rangle_{\dot{H}^s} = -\sum_{k=1}^d \int_{B(0,n)} |\xi|^{2s} \xi_k^2 |\widehat{u}_n(t,\xi)|^2 d\xi = -\sum_{k=1}^d \langle \xi_k \widehat{u}_n, \xi_k \widehat{u}_n\rangle_{L^2(B(0,n),|\xi|^{2s}d\xi)}$$
$$= \sum_{k=1}^d \langle \xi_k \widehat{u}_n, \xi_k \widehat{u}_n\rangle_{L^2(\mathbb{R}^d,|\xi|^{2s}d\xi)} = \|\nabla u_n\|_{\dot{H}^s}^2,$$

we obtain

$$\frac{1}{2}\frac{d}{dt}\|u_n\|_{\dot{H}^s}^2 + \nu \|\nabla u_n\|_{\dot{H}^s}^2 = \langle \mathbf{P}_n f_n, u_n\rangle_{\dot{H}^s}$$

s.t. after integration we obtain

$$\frac{1}{2}\|u_n(t)\|_{\dot{H}^s}^2 + \nu \int_0^t \|\nabla u_n(t')\|_{\dot{H}^s}^2 dt' = \frac{1}{2}\|\mathbf{P}_n u_0\|_{\dot{H}^s}^2 + \int_0^t \langle \mathbf{P}_n f_n(t'), u_n(t')\rangle_{\dot{H}^s} dt'. \tag{5.9}$$

The difference $u_n - u_{n+\ell}$ solves

$$\begin{cases} (u_n - u_{n+\ell})_t - \nu \mathbf{P}_{n+\ell} \triangle (u_n - u_{n+\ell}) = \mathbf{P}_n f_n - \mathbf{P}_{n+\ell} f_{n+\ell} \\ u_n(0) - u_{n+\ell}(0) = (\mathbf{P}_n - \mathbf{P}_{n+\ell}) u_0 \end{cases}$$

Then, like for (5.9) we get

$$
\begin{aligned}
&\frac{1}{2}\|u_n(t)-u_{n+\ell}(t)\|^2_{\dot H^s}+\not{2}\frac{\nu}{2}\int_0^t\|\nabla(u_n-u_{n+\ell})(t')\|^2_{\dot H^s}dt'=\\
&=\frac{1}{2}\|(\mathbf{P}_n-\mathbf{P}_{n+\ell})u_0\|^2_{\dot H^s}+\int_0^t\langle \mathbf{P}_nf_n(t')-\mathbf{P}_{n+\ell}f_{n+\ell}(t'),(u_n-u_{n+\ell})(t')\rangle_{\dot H^s}dt'\\
&\le\frac{1}{2}\|(\mathbf{P}_n-\mathbf{P}_{n+\ell})u_0\|^2_{\dot H^s}+\int_0^t\|\mathbf{P}_nf_n(t')-\mathbf{P}_{n+\ell}f_{n+\ell}(t')\|_{\dot H^{s-1}}\|\nabla(u_n-u_{n+\ell})(t')\|_{\dot H^s}dt'\\
&\le\frac{1}{2}\|(\mathbf{P}_n-\mathbf{P}_{n+\ell})u_0\|^2_{\dot H^s}+\frac{1}{2\nu}\int_0^t\|\mathbf{P}_nf_n(t')-\mathbf{P}_{n+\ell}f_{n+\ell}(t')\|^2_{\dot H^{s-1}}dt'+\cancel{\frac{\nu}{2}\int_0^t\|\nabla(u_n-u_{n+\ell})(t')\|^2_{\dot H^s}dt'}.
\end{aligned}
$$

Hence

$$
\begin{aligned}
&\|u_n(t)-u_{n+\ell}(t)\|^2_{\dot H^s}+\nu\int_0^t\|\nabla(u_n-u_{n+\ell})(s)\|^2_{\dot H^s}ds\\
&\le\|(\mathbf{P}_n-\mathbf{P}_{n+\ell})u_0\|^2_{\dot H^s}+\frac{1}{\nu}\int_0^t\|\mathbf{P}_nf_n(s)-\mathbf{P}_{n+\ell}f_{n+\ell}(s)\|^2_{\dot H^{s-1}}ds.
\end{aligned}
$$

Since $f_n \overset{n\to\infty}{\to} f$ in $L^2([0,T],(\dot H^{s-1}(\mathbb{R}^d))^d)$ implies also $\mathbf{P}_nf_n \overset{n\to\infty}{\to} f$ therein, the last inequality implies that (u_n) is Cauchy in $C([0,T],(\dot H^s(\mathbb{R}^d))^d))$ and (∇u_n) is Cauchy in $L^2([0,T],(\dot H^s(\mathbb{R}^d))^d))$. Let u be the limit. Notice that u satisfies (5.2) and (5.6), and so obviously also (5.3).
Taking the limit in (5.9) we see that u satisfies the energy equality(5.5).
Next, we check that u is a weak solution of (5.1) in the sense of Def. 5.1. We apply $\langle .,\Psi(t)\rangle_{L^2}$ to (5.8) with $\Psi\in C_c^\infty([0,\infty)\times\mathbb{R}^d,\mathbb{R}^d)$. Then we have

$$
\frac{d}{dt}\langle u_n,\Psi\rangle_{L^2}=\nu\langle\triangle u_n,\Psi\rangle_{L^2}+\langle\mathbf{P}_nf_n,\Psi\rangle_{L^2}+\langle u_n,\partial_t\Psi\rangle_{L^2}.
$$

Integrating we have

$$
\begin{aligned}
\langle u_n(t),\Psi(t)\rangle_{L^2}&=\langle\mathbf{P}_nu_0,\Psi(0)\rangle_{L^2}-\nu\int_0^t\langle u_n(t'),\triangle\Psi(t')\rangle_{L^2}dt'\\
&+\int_0^t\langle\mathbf{P}_nf_n(t'),\Psi(t')\rangle_{L^2}dt'+\int_0^t\langle u_n(t'),\partial_t\Psi(t')\rangle_{L^2}dt'.
\end{aligned}
$$

Taking the limit for $n\to\infty$ we get

$$
\langle u(t),\Psi(t)\rangle_{L^2}=\langle u_0,\Psi(0)\rangle_{L^2}-\nu\int_0^t\langle u(t'),\triangle\Psi(t')\rangle_{L^2}dt'+\int_0^t\langle f(t'),\Psi(t')\rangle_{L^2}dt'+\int_0^t\langle u(t'),\partial_t\Psi(t')\rangle_{L^2}dt'.
$$

which yields (11.5). Hence u is a weak solution of (5.1) in the sense of Def. 5.1.
Next, we prove the Duhamel formula (10.8). Applying the Fourier transform to (5.8)

$$
\begin{cases}\partial_t\widehat{u}_n(t,\xi)+\nu\chi_{|\xi|\le n}|\xi|^2\widehat{u}_n(t,\xi)=\chi_{|\xi|\le n}\widehat{f}_n(t,\xi)\\ \widehat{u_n}(0,\xi)=\chi_{|\xi|\le n}\widehat{u}_0(\xi)\end{cases}\tag{5.10}
$$

Notice that $\text{supp}\widehat{u}_n(t,\cdot) \subseteq \{|\xi| \le n\}$ so that $\chi_{|\xi|\le n}|\xi|^2\widehat{u}_n(t,\xi) = |\xi|^2\widehat{u}_n(t,\xi)$. Then, by the variation of parameters formula

$$\widehat{u}_n(t,\xi) = e^{-t\nu|\xi|^2}\chi_{|\xi|\le n}\widehat{u}_0(\xi) + \int_0^t e^{-(t-t')\nu|\xi|^2}\chi_{|\xi|\le n}\widehat{f}_n(t',\xi)dt'. \tag{5.11}$$

Now we know

$$\begin{aligned}
&\widehat{u}_n(t,\xi) \overset{n\to\infty}{\to} \widehat{u}(t,\xi) \text{ in } C([0,T], L^2(\mathbb{R}^d, |\xi|^{2s}d\xi))\\
&\chi_{|\xi|\le n}\widehat{u}_0(\xi) \overset{n\to\infty}{\to} \widehat{u}_0(\xi) \text{ in } L^2(\mathbb{R}^d, |\xi|^{2s}d\xi),\\
&\chi_{|\xi|\le n}\widehat{f}_n(t',\xi) \overset{n\to\infty}{\to} \widehat{f}(t',\xi) \text{ in } L^2([0,T]\times\mathbb{R}^d, |\xi|^{2(s-1)}dtd\xi)
\end{aligned}$$

Notice that

$$\mathbf{T}g(t,\xi) := \int_0^t e^{-(t-t')\nu|\xi|^2}g(t',\xi)dt'$$

is a bounded operator from $L^2([0,T]\times\mathbb{R}^d, |\xi|^{2(s-1)}dtd\xi)$ into $L^\infty([0,T], L^2(\mathbb{R}^d, |\xi|^{2s}d\xi)$. Indeed for $t\in[0,T]$ and fixed $\xi\in\mathbb{R}^d$ and for $g\in C_c([0,T]\times(\mathbb{R}^d\backslash\{0\}))$

$$|\mathbf{T}g(t,\xi)| \le (\int_0^t e^{-2(t-t')\nu|\xi|^2}dt')^{\frac{1}{2}}(\int_0^t |g(t',\xi)|^2dt')^{\frac{1}{2}} \le \frac{1}{\sqrt{2\nu}|\xi|}(\int_0^t |g(t',\xi)|^2dt')^{\frac{1}{2}}$$

and so

$$\int_{\mathbb{R}^d}|\xi|^{2s}|\mathbf{T}g(t,\xi)|^2d\xi \le \frac{1}{2\nu}\int_{[0,T]\times\mathbb{R}^d}|\xi|^{2(s-1)}|g(t',\xi)|^2dt'd\xi.$$

This implies

$$\|\mathbf{T}g\|_{L^\infty([0,T],L^2(\mathbb{R}^d,|\xi|^{2s}d\xi)} \le \sqrt{1/2\nu}\|g\|_{L^2([0,T]\times\mathbb{R}^d,|\xi|^{2(s-1)}dtd\xi)}.$$

Since $C_c([0,T]\times(\mathbb{R}^d\backslash\{0\}))$ is dense in $L^2([0,T]\times\mathbb{R}^d, |\xi|^{2(s-1)}dtd\xi)$ a well defined bounded operator remains defined. Taking the limit for $n\to\infty$ in (5.11) all terms converge in $L^\infty([0,T], L^2(\mathbb{R}^d, |\xi|^{2s}d\xi))$ to the corresponding terms of

$$\widehat{u}(t,\xi) = e^{-t\nu|\xi|^2}\widehat{u}_0(\xi) + \int_0^t e^{-(t-t')\nu|\xi|^2}\widehat{f}(t',\xi)dt'.$$

□

Remark 5.3. Notice that applying the Fourier transform to (10.8) we get

$$u(t) = e^{t\nu\triangle}u_0 + \int_0^t e^{(t-t')\nu\triangle}f(t')dt'. \tag{5.12}$$

The following theorem yields additional estimates.

Theorem 5.4. *Let f be like in Theorem 5.2 and consider the corresponding solution*

$$u \in C([0,T], \dot{H}^s), \quad \nabla u \in L^2([0,T], \dot{H}^s).$$

Then, additionally, we have

$$\|u(t)\|_{\dot{H}^{s+\frac{2}{p}}} \in L^p([0,T], \mathbb{R}) \text{ for any } p \geq 2. \tag{5.13}$$

Moreover we have

$$\begin{aligned} V(t) &:= \left(\int_{\mathbb{R}^d} |\xi|^{2s} \left(\sup_{0 \leq t' \leq t} |\widehat{u}(t',\xi)| \right)^2 d\xi \right)^{\frac{1}{2}} \leq \|u_0\|_{\dot{H}^s} + \frac{1}{(2\nu)^{\frac{1}{2}}} \|f\|_{L^2([0,t],\dot{H}^{s-1})} \ ; \\ \| \|u\|_{\dot{H}^{s+\frac{2}{p}}} \|_{L^p(0,T)} &\leq \nu^{-\frac{1}{p}} \left(\|u_0\|_{\dot{H}^s} + \nu^{-\frac{1}{2}} \|f\|_{L^2([0,T],\dot{H}^{s-1})} \right) \ . \end{aligned} \tag{5.14}$$

Proof. From the Duhamel formula (10.8) and the previous computation

$$|\widehat{u}(t,\xi)| \leq e^{-t\nu|\xi|^2} |\widehat{u}_0(\xi)| + \frac{1}{\sqrt{2\nu}|\xi|} \|\widehat{f}(\cdot,\xi)\|_{L^2(0,t)}.$$

so that

$$|\xi|^s \sup_{0 \leq t' \leq t} |\widehat{u}(t',\xi)| \leq |\xi|^s |\widehat{u}_0(\xi)| + |\xi|^s \frac{1}{\sqrt{2\nu}|\xi|} \|\widehat{f}(\cdot,\xi)\|_{L^2(0,t)}.$$

Taking the $L^2(\mathbb{R}^d, d\xi)$ norm we get

$$V(t) \leq \|u_0(\xi)\|_{L^2(\mathbb{R}^d,|\xi|^{2s}d\xi)} + \frac{1}{\sqrt{2\nu}} \|\widehat{f}\|_{L^2((0,t),L^2(\mathbb{R}^d,|\xi|^{2(s-1)}d\xi))}.$$

and this yields the 1st line in (5.14).
To get the 2nd line in (5.14), from the energy estimate (5.5) we obtain

$$\begin{aligned} \|u(t)\|^2_{\dot{H}^s} + \cancel{2}\nu \int_0^t \|\nabla u(t')\|^2_{\dot{H}^s} dt' &\leq \|u_0\|^2_{\dot{H}^s} + 2\int_0^t \frac{1}{\sqrt{\nu}} \|f(t')\|_{\dot{H}^{s-1}} \sqrt{\nu} \|\nabla u(t')\|_{\dot{H}^s} dt' \\ &\leq \|u_0\|^2_{\dot{H}^s} + \cancel{\nu \int_0^t \|\nabla u(t')\|^2_{\dot{H}^s} dt'} + \frac{1}{\nu} \int_0^t \|f(t')\|^2_{\dot{H}^{s-1}} dt'. \end{aligned}$$

This yields

$$\|u(t)\|^2_{\dot{H}^s} + \nu \int_0^t \|\nabla u(t')\|^2_{\dot{H}^s} dt' \leq \|u_0\|^2_{\dot{H}^s} + \frac{1}{\nu} \int_0^t \|f(t')\|^2_{\dot{H}^{s-1}} dt'.$$

and hence

$$\begin{aligned} &\|u\|_{L^\infty([0,T],\dot{H}^s)} \leq \|u_0\|_{\dot{H}^s} + \nu^{-\frac{1}{2}} \|f\|_{L^2([0,T],\dot{H}^s)} \\ &\| \|u\|_{\dot{H}^{s+1}} \|_{L^2(0,T)} \leq \nu^{-\frac{1}{2}} \left(\|u_0\|_{\dot{H}^s} + \nu^{-\frac{1}{2}} \|f\|_{L^2([0,T],\dot{H}^s)} \right). \end{aligned}$$

So by the interpolation of Sobolev norms Lemma 4.1 for $2 < p < \infty$

$$\|\|u\|_{\dot H^{s+\frac{2}{p}}}\|_{L^p(0,T)} \le \|\|u\|_{\dot H^s}^{1-\frac{2}{p}}\|\nabla u\|_{\dot H^s}^{\frac{2}{p}}\|_{L^p(0,T)} \le \|u\|_{L^\infty([0,T],\dot H^s)}^{1-\frac{2}{p}}\|\|\nabla u\|_{\dot H^s}^{\frac{2}{p}}\|_{L^p(0,T)}$$
$$= \|u\|_{L^\infty([0,T],\dot H^s)}^{1-\frac{2}{p}}\|\nabla u\|_{L^2([0,T],\dot H^s)}^{\frac{2}{p}} \le \nu^{-\frac{1}{p}}\left(\|u_0\|_{\dot H^s} + \nu^{-\frac{1}{2}}\|f\|_{L^2([0,T],\dot H^s)}\right).$$

□

6 The Navier Stokes equation

We will only deal with the Incompressible Navier Stokes (NS) equation:

$$\begin{cases} u_t + u\cdot\nabla u - \nu\triangle u = -\nabla p \\ \nabla\cdot u = 0 \qquad (t,x)\in[0,\infty)\times\mathbb{R}^d \\ u(0,x) = u_0(x) \end{cases} \tag{6.1}$$

where $u : [0,\infty)\times\mathbb{R}^d \to \mathbb{R}^d$ with $u = \sum_{j=1}^d u^j e_j$ with e_j the standard basis of $\mathbb{R}^d$,

$$\triangle := \sum_{j=1}^d \frac{\partial^2}{\partial x_j^2}\,,\ \nabla\cdot u = \sum_{j=1}^d \frac{\partial}{\partial x_j}u^j\,,\ u\cdot\nabla v = \sum_{j=1}^d u_j\frac{\partial}{\partial x_j}v.$$

Here $\nu > 0$ is a fixed constant. We could normalize $\nu = 1$. p is the pressure and its function is simply to absorb the divergence part of the l.h.s. of (6.1).

We can write

$$u\cdot\nabla u = \mathrm{div}(u\otimes u) \text{ for } \mathrm{div}(u\otimes v)^j := \sum_{k=1}^d \partial_k(u^k v^j) \text{ since} \tag{6.2}$$
$$\mathrm{div}(u\otimes u)^j = \sum_{k=1}^d \partial_k(u^k u^j) = \sum_{k=1}^d u^k\partial_k u^j + u^j\underbrace{\mathrm{div}u}_{0} = u\cdot\nabla u^j$$

So we rewrite (6.1) and

$$\begin{cases} u_t + \mathrm{div}(u\otimes u) - \nu\triangle u = -\nabla p \\ \nabla\cdot u = 0 \qquad (t,x)\in[0,\infty)\times\mathbb{R}^d \\ u(0,x) = u_0(x) \end{cases} \tag{6.3}$$

Definition 6.1 (Weak solutions)**.** Let u_0 be in $L^2(\mathbb{R}^d)$. A vector field $u \in L^2_{loc}([0,\infty)\times\mathbb{R}^d)$ which is weakly continuous as a function from $[0,\infty)$ to $(L^2(\mathbb{R}^d))^d$ is a weak solution of (6.3) if for $\Psi \in C_c^\infty([0,\infty)\times\mathbb{R}^d,\mathbb{R}^d)$ with $\mathrm{div}\Psi = 0$ we have

$$\begin{aligned}\langle u(t),\Psi(t)\rangle_{L^2} = \int_0^t \big(\nu\langle u(t'),\triangle\Psi(t')\rangle_{L^2} + \langle u(t'),\partial_t\Psi(t')\rangle_{L^2} \\ -\langle \mathrm{div}(u\otimes u)(t'),\Psi(t')\rangle_{L^2}\big)\,dt' + \langle u_0,\Psi(0)\rangle_{L^2}.\end{aligned} \tag{6.4}$$

Notice that formally (6.4) is obtained from (6.3) writing

$$\int_0^t \int_{\mathbb{R}^d} (u_t + \operatorname{div}(u \otimes u) - \nu \triangle u) \cdot \Psi = - \int_0^t \int_{\mathbb{R}^d} \nabla p \cdot \Psi = \int_0^t \int_{\mathbb{R}^d} p \underbrace{\nabla \cdot \Psi}_{0}.$$

So integrating by parts (which is formal if u is not sufficiently regular) we have

$$\begin{aligned} 0 &= \int_{\mathbb{R}^d} u \cdot \Psi \Big|_0^t - \int_0^t \int_{\mathbb{R}^d} u \cdot \partial_t \Psi + \int_0^t \int_{\mathbb{R}^d} \partial_k (u^j u^k) \Psi^j - \nu \int_0^t \int_{\mathbb{R}^d} u \cdot \triangle \Psi \\ &= \int_{\mathbb{R}^d} u \cdot \Psi \Big|_0^t - \int_0^t \int_{\mathbb{R}^d} u \cdot \partial_t \Psi - \int_0^t \int_{\mathbb{R}^d} u^j u^k \partial_k \Psi^j - \nu \int_0^t \int_{\mathbb{R}^d} u \cdot \triangle \Psi \end{aligned}$$

which gives the desired result. In particular, (6.3) implies (6.4) when u is regular.
But the opposite is also true, and when u is regular (6.4) implies (6.3). Indeed, suppose that u is regular and that it satisfies (6.4) for all the Ψ as in Def. 6.1. Then

$$\begin{aligned} &\int_{\mathbb{R}^d} u(t,x) \cdot \Psi(t,x) dx - \int_{\mathbb{R}^d} u_0(x) \cdot \Psi(0,x) dx = \int_0^t \int_{\mathbb{R}^d} \left(\nu u \cdot \triangle \Psi + u \otimes u : \nabla \Psi + u \partial_t \Psi \right) (t', x) dx dt' \\ &= \int_0^t \int_{\mathbb{R}^d} (\nu \triangle u - \operatorname{div}(u \otimes u) - \partial_t u) \cdot \Psi + \int_{\mathbb{R}^d} u(t,x) \cdot \Psi(t,x) dx - \int_{\mathbb{R}^d} u(0,x) \cdot \Psi(0,x) dx. \end{aligned}$$

Hence we get

$$\int_{\mathbb{R}^d} u_0(x) \cdot \Psi(0,x) dx = \int_0^t \int_{\mathbb{R}^d} (\partial_t u - \nu \triangle u + \operatorname{div}(u \otimes u)) \cdot \Psi + \int_{\mathbb{R}^d} u(0,x) \cdot \Psi(0,x) dx.$$

Taking $\Psi = \varphi(t)\psi(x)$ with $\varphi \in C_c^\infty((0,T), \mathbb{R})$ and $\psi \in C_c^\infty(\mathbb{R}^d, \mathbb{R}^d)$ and divergence free, we conclude that

$$\int_0^t dt' \varphi(t) \int_{\mathbb{R}^d} \left[(\partial_t u - \nu \triangle u + \operatorname{div}(u \otimes u)) \cdot \psi(x) \right] dx.$$

This implies that for all t

$$\langle \nu \triangle u - \operatorname{div}(u \otimes u) - \partial_t u, \psi \rangle_{L^2(\mathbb{R}^d)} = 0$$

for any t and for any divergence free vector field $\psi \in C_c^\infty(\mathbb{R}^d, \mathbb{R}^d)$. Formally, this implies that the above holds for $\psi = \mathbb{P}\Theta$ for any vector field $\Theta \in C_c^\infty(\mathbb{R}^d, \mathbb{R}^d)$. Then, by $\mathbb{P}^* = \mathbb{P}$, we conclude that

$$\langle \mathbb{P}(\nu \triangle u - \operatorname{div}(u \otimes u) - \partial_t u), \Theta \rangle_{L^2(\mathbb{R}^d)} = 0 \text{ for all } \Theta \in C_c^\infty(\mathbb{R}^d, \mathbb{R}^d).$$

This implies

$$\mathbb{P}(\nu \triangle u - \operatorname{div}(u \otimes u) - \partial_t u) = 0 \Rightarrow u_t + u \cdot \nabla u - \nu \triangle u = -\nabla p$$

for some p, see Lemma 2.5.

Then we get

$$\int_{\mathbb{R}^d} u_0(x)\cdot\Psi(0,x)dx = \int_{\mathbb{R}^d} u(0,x)\cdot\Psi(0,x)dx$$

and so $u(0,x) = u_0(x)$.

Let us now formally take the inner product of the first line of (6.1) with u and integrate in $\mathbb{R}^d$

$$\frac{1}{2}\frac{d}{dt}\|u\|_{L^2}^2 + \langle u\cdot\nabla u, u\rangle_{L^2} - \nu\langle\triangle u, u\rangle_{L^2} = -\langle\nabla p, u\rangle_{L^2}$$

We have, summing on repeated indexes,

$\langle u\cdot\nabla u, u\rangle_{L^2} = \int_{\mathbb{R}^d} u^j u^k \partial_j u^k dx = 2^{-1}\int_{\mathbb{R}^d} u^j\partial_j(u^k u^k)^2 dx = -2^{-1}\int_{\mathbb{R}^d} |u|^2 \text{div} u \; dx = 0$ and

$\langle\nabla p, u\rangle_{L^2} = \int_{\mathbb{R}^d} u^j\partial_j p dx = -\int_{\mathbb{R}^d} p \text{div} u \; dx = 0.$

So, formally (rigorously if u is regular and we can integrate by parts), we get

$$\frac{1}{2}\frac{d}{dt}\|u\|_{L^2}^2 + \nu\|\nabla u\|_{L^2}^2 = 0$$

This in particular yields the following *energy equality*

$$\|u(t)\|_{L^2(\mathbb{R}^d)}^2 + 2\nu\int_0^t \|\nabla u(t')\|_{L^2(\mathbb{R}^d)}^2 dt' = \|u_0\|_{L^2(\mathbb{R}^d)}^2. \tag{6.5}$$

Theorem 6.2 (Leray). *Let $u_0 \in L^2(\mathbb{R}^d)$ for $d = 2, 3$ be divergence free. Then* (6.3) *admits a weak solution with $u(t) \in L^\infty(\mathbb{R}_+, L^2) \cap L^2_{loc}(\mathbb{R}_+, H^1)$ such that the following energy inequality holds:*

$$\|u(t)\|_{L^2(\mathbb{R}^d)}^2 + 2\nu\int_0^t \|\nabla u(t')\|_{L^2(\mathbb{R}^d)}^2 dt' \le \|u_0\|_{L^2(\mathbb{R}^d)}^2. \tag{6.6}$$

We will also see the following.

Theorem 6.3 (Case $d = 2$). *When $d = 2$ the solution in Theorem 6.2 is unique, it satisfies* (6.5) *and $u(t) \in C^0([0,\infty), L^2)$.*

Notice that if we apply formally the operator $\mathbb{P}$ to equation (6.3) we obtain formally

$$\begin{cases} u_t - \nu\triangle u = \mathcal{Q}_{NS}(u,u) \\ u(0,x) = u_0(x) \end{cases} (t,x) \in [0,\infty)\times\mathbb{R}^d \tag{6.7}$$

where we set

$$\mathcal{Q}_{NS}(u,v) := -\frac{1}{2}\mathbb{P}(\text{div}(u\otimes v)) - \frac{1}{2}\mathbb{P}(\text{div}(v\otimes u)). \tag{6.8}$$

Here notice that

$$\mathbb{P}(\mathrm{div}(u \otimes v))^j = \sum_{l=1}^{d} \partial_l \left((u^l v^j) - \frac{1}{\triangle} \sum_{k=1}^{d} \partial_j \partial_k (u^l v^k) \right). \tag{6.9}$$

Before starting the proof of Theorem 6.2 we need some preliminary results on 1st order ODE's in Banach spaces.

Definition 6.4. Given a Banach space X a function $F : X \to X$ is locally Lipschitz if for any $M > 0$ $\exists$ $L(M) \in (0, +\infty)$ s.t.

$$\|F(x) - F(y)\| \le L(M)\|x - y\| \text{ for all } x, y \text{ with } \|x\| \le M \text{ and } \|y\| \le M. \tag{6.10}$$

Now consider the system

$$\dot{u} = F(u)\,, \quad u(0) = x \tag{6.11}$$

which we write in integral form as

$$u(t) = x + \int_0^t F(u(s))ds. \tag{6.12}$$

Proposition 6.5. *Let F be as in Definition 6.4. Then for any $M > 0$, for T_M defined by*

$$T_M := \frac{1}{2L(2M + \|F(0)\|) + 2}. \tag{6.13}$$

and for any $x \in X$ with $\|x\| \le M$ there is a unique solution $u \in C^0([0, T_M], X)$ of (6.12).

Proof. Set $K = 2M + \|F(0)\|$ and

$$E = \{u \in C^0([0, T_M], X) : \|u(t)\| \le K \text{ for all } t \in [0, T_M]\}$$

with distance $d_E(u, v) := \sup_{0 \le t \le T_M} \|u(t) - v(t)\|$. (E, d_E) is a complete metric space. Next consider the map $u \in E \to \Phi_u$

$$\Phi_u(t) := x + \int_0^t F(u(s))ds \text{ for all } t \in [0, T_M].$$

By $T_M = \frac{1}{2(L(K)+1)}$ for all $t \in [0, T_M]$ we have

$$\begin{aligned} &\|F(u(t))\| \le \|F(0)\| + \|F(u(t)) - F(0)\| \le \|F(0)\| + KL(K) \\ &= \|F(0)\| + (2M + \|F(0)\|)L(K) \le (M + \|F(0)\|)\ 2\ (L(K) + 1) = \frac{M + \|F(0)\|}{T_M} \end{aligned}$$

So for $t \in [0, T_M]$ we have

$$\|\Phi_u(t)\| \le M + t\frac{M + \|F(0)\|}{T_M} \le 2M + \|F(0)\| = K$$

and so $\Phi_u \in E$.
For $u, v \in E$ we have

$$\|\Phi_u(t) - \Phi_v(t)\| \leq \int_0^t \|F(u(s)) - F(v(s))\| ds \leq T_M L(K) \|u - v\|_{L^\infty([0,T],X)} = T_M L(K) d_E(u, v).$$

So by $T_M L(K) < T_M(L(K) + 1) = 2^{-1}$

$$d_E(\Phi_u, \Phi_v) \leq 2^{-1} d_E(u, v)$$

Hence $u \to \Phi_u$ is a contraction in E and so it has exactly one fixed point.

□

We have the following application of Gronwall's inequality.

Lemma 6.6. *Let $T > 0$, $x \in X$ and let $u, v \in C^0([0,T], X)$ solve (6.12) then $u = v$.*

Proof. Let $M = \max_{0 \leq t \leq T}\{\|u(t)\|, \|v(t)\|\}$. Then

$$\|u(t) - v(t)\| \leq \int_0^t \|F(u(s)) - F(v(s))\| ds \leq L(M) \int_0^t \|u(s) - v(s)\| ds$$

and apply Gronwall's inequality. □

It remains defined a function $T : X \to (0, \infty]$ where for any $x \in X$

$$T(x) = \sup\{T > 0 : \exists u \in C^0([0,T), X) \text{ solution of (6.12) }\}$$

and the interval $[0, T(x))$ is the largest (positive) half open interval of existence of the (unique, by Lemma 13.6) solution of (6.12).

Theorem 6.7. *We have, for $u(t)$ the corresponding solution in $C([0, T(x)), X)$,*

$$2L(\|F(0)\| + 2\|u(t)\|) \geq \frac{1}{T(x) - t} - 2 \tag{6.14}$$

for all $t \in [0, T(x))$. We have the alternatives

(1) either $T(x) = +\infty$;

(2) or if $T(x) < +\infty$ then $\lim_{t \nearrow T(x)} \|u(t)\| = +\infty$.

Proof. First of all it is obvious that if $T(x) < +\infty$ then by (13.10)

$$\lim_{t \nearrow T(x)} L(\|F(0)\| + 2\|u(t)\|) = +\infty \Rightarrow \lim_{t \nearrow T(x)} \|u(t)\| = +\infty$$

where the implication follows from the fact that $M \to L(M)$ is an increasing function.

We are left with the proof of (13.10), which is clearly true if $T(x) = \infty$. Now suppose that $T(x) < \infty$ and that (13.10) is false. This means that there exists a $t \in [0, T(x))$ with

$$\frac{1}{T_M} - 2 = 2L(\|F(0)\| + 2\|u(t)\|) < \frac{1}{T(x) - t} - 2 \Rightarrow T(x) - t < T_M$$

for $M = \|u(t)\|$, where we recall $T_M := \frac{1}{2L(2M + \|F(0)\|) + 2}$ in (6.13). Consider now $v \in C^0([0, T_M], X)$ the solution of

$$v(s) = u(t) + \int_0^s F(v(s'))ds' \text{ for all } s \in [0, T_M].$$

which exists by the previous Proposition 6.5. Then define

$$w(s) := \begin{cases} u(s) \text{ for } s \in [0, t] \\ v(s - t) \text{ for } s \in [t, t + T_M]. \end{cases}$$

We claim that $w \in C^0([0, t + T_M], X)$ is a solution of (6.12). In $[0, t]$ this is obvious since in $w = u$ in $[0, t]$ and $u \in C^0([0, t], X)$ is a solution of (6.12). Let now $s \in (t, t + T_M]$. We have

$$\begin{aligned} w(s) = v(s - t) = u(t) + \int_0^{s-t} F(v(s'))ds' \\ = x + \int_0^t F(u(s'))ds' + \int_0^{s-t} F(v(s'))ds' \\ = x + \int_0^t F(\underbrace{u(s')}_{w(s')})ds' + \int_t^s F(\underbrace{v(s' - t)}_{w(s')})ds' \\ = x + \int_0^s F(w(s'))ds. \end{aligned}$$

□

6.1 Proof of Theorem 6.2

We will need the following elementary lemma.

Lemma 6.8. *Let $d = 2, 3$. Then the trilinear form*

$$(u, v, \varphi) \in (C_c^\infty(\mathbb{R}^d))^d \times (C_c^\infty(\mathbb{R}^d))^d \times (C_c^\infty(\mathbb{R}^d))^d \to \langle div(u \otimes v), \varphi \rangle_{L^2} \in \mathbb{R} \tag{6.15}$$

extends into a unique bounded trilinear form $(H^1(\mathbb{R}^d))^d \times (H^1(\mathbb{R}^d))^d \times (H^1(\mathbb{R}^d))^d$ which satisfies for a fixed C

$$\langle div(u \otimes v), \varphi \rangle_{L^2} \leq C \|\nabla u\|_{L^2}^{\frac{d}{4}} \|\nabla v\|_{L^2}^{\frac{d}{4}} \|u\|_{L^2}^{1-\frac{d}{4}} \|v\|_{L^2}^{1-\frac{d}{4}} \|\nabla \varphi\|_{L^2} \tag{6.16}$$

If furthermore $div\ u = 0$ then

$$\langle div(u \otimes v), v \rangle_{L^2} = 0. \tag{6.17}$$

Proof. Recall that from (6.2) we have $\mathrm{div}(u \otimes v)^j := \sum_{k=1}^d \partial_k(u^k v^j)$. Then for fields like in (6.15) we have

$$\langle \mathrm{div}(u\otimes v), \varphi\rangle_{L^2} = \sum_{j=1}^d \langle \mathrm{div}(u\otimes v)^j, \varphi^j\rangle_{L^2} = \sum_{j=1}^d \langle \sum_{k=1}^d \partial_k(u^k v^j), \varphi^j\rangle_{L^2} = -\sum_{j=1}^d\sum_{k=1}^d \langle u^k v^j, \partial_k \varphi^j\rangle_{L^2}.$$

Now the r.h.s. can be bounded by

$$|\langle u^k v^j, \partial_k \varphi^j\rangle_{L^2}| \leq \|u^k v^j\|_{L^2}\|\nabla\varphi\|_{L^2} \leq \|u^k\|_{L^4}\|v^j\|_{L^4}\|\nabla\varphi\|_{L^2}.$$

Finally, we apply Gagliardo-Nirenberg inequality writing

$$\|u^k\|_{L^4} \leq C\|\nabla u^k\|_{L^2}^{\frac{d}{4}}\|u^k\|_{L^2}^{1-\frac{d}{4}}.$$

The same equality holds for v^j. Then we obtain (6.16), obviously with a different C. This implies that that the form in (6.15) is continuous, and by density of $C_c^\infty(\mathbb{R}^d)$ in $H^1(\mathbb{R}^d)$ it extends in a unique way.
Next, we write for $\varphi = v$

$$\begin{aligned}
\langle \mathrm{div}(u\otimes v), v\rangle_{L^2} &= -\sum_{j=1}^d\sum_{k=1}^d \langle u^k v^j, \partial_k v^j\rangle_{L^2} \\
&= -2^{-1}\sum_{j=1}^d\sum_{k=1}^d \langle u^k, \partial_k (v^j)^2\rangle_{L^2} = 2^{-1}\sum_{j=1}^d \langle (\mathrm{div} u) v^j, v^j\rangle_{L^2} = 0.
\end{aligned}$$

Notice that this formal computation (the Leibnitz rule used for the 2nd equality requires some explaining) is certainly rigorous for $v \in (C_c^\infty(\mathbb{R}^d))^d$. On the other hand inequality (6.16) yields (6.17) by a density argument also for $v \in (H^1(\mathbb{R}^d))^d$. ☐

Remark 6.9. Notice that $u, v \in (H^1(\mathbb{R}^d))^d$ implies $\mathrm{div}(u\otimes v) \in (L^1(\mathbb{R}^d))^d$. Indeed we have

$$\mathrm{div}(u\otimes v)^j = \sum_{k=1}^d \partial_k(u^k v^j) = \sum_{k=1}^d (v^j\partial_k u^k + u^k \partial_k v^j) \tag{6.18}$$

where the above product rule can be proved by taking sequences $(C_c^\infty(\mathbb{R}^d))^d \ni u_n \overset{n\to\infty}{\to} u$ in $(H^1(\mathbb{R}^d))^d$ and $(C_c^\infty(\mathbb{R}^d))^d \ni v_n \overset{n\to\infty}{\to} v$ in $(H^1(\mathbb{R}^d))^d$. Then clearly for $\psi \in (\mathcal{S}^\infty(\mathbb{R}^d))^d$ summing on double indexes

$$\begin{aligned}
&\langle \partial_k(u^k v^j), \psi^j\rangle = -\langle u^k v^j, \partial_k\psi^j\rangle = -\lim_{n\to\infty}\langle u_n^k v_n^j, \partial_k \psi^j\rangle \\
&= \lim_{n\to\infty}\left(\langle v_n^j \partial_k u_n^k, \psi^j + u_n^k\partial_k v_n^j, \psi^j\rangle\right) = \langle v^j\partial_k u^k + u^k\partial_k v^j, \psi^j\rangle
\end{aligned}$$

and this yields (6.18).

Hence $\mathfrak{F} := \mathcal{F}(\mathrm{div}(u \otimes v)) \in (L^\infty(\mathbb{R}^d))^d \subset (L^1_{loc}(\mathbb{R}^d))^d$. Furthermore, (6.16) implies that $\mathfrak{F} \in (L^2(\mathbb{R}^d, |\xi|^{-2} d\xi))^d$. Indeed the bilinear map

$$\langle \cdot, \cdot \rangle_{L^2(\mathbb{R}^d, d\xi)} : L^2(\mathbb{R}^d, |\xi|^{-2} d\xi) \times L^2(\mathbb{R}^d, |\xi|^2 d\xi) \to \mathbb{R}$$

can be used to define an embedding

$$L^2(\mathbb{R}^d, |\xi|^{-2} d\xi) \hookrightarrow (L^2(\mathbb{R}^d, |\xi|^2 d\xi))'$$

by $f \to \langle f, \cdot \rangle_{L^2(\mathbb{R}^d, d\xi)}$. Furthermore we have the commutative diagram

$$\begin{array}{ccc} L^2(\mathbb{R}^d, |\xi|^{-2} d\xi) & \overset{f \to \langle f, \cdot \rangle_{L^2(\mathbb{R}^d, d\xi)}}{\to} & (L^2(\mathbb{R}^d, |\xi|^2 d\xi))' \\ f \to |\xi|^{-1} f \downarrow & & \uparrow \\ L^2(\mathbb{R}^d, d\xi) & \overset{h \to \langle h, \cdot \rangle_{L^2(\mathbb{R}^d, d\xi)}}{\to} & (L^2(\mathbb{R}^d, d\xi))' \end{array} \tag{6.19}$$

where the $\uparrow$ is the map $(L^2(\mathbb{R}^d, d\xi))' \to (L^2(\mathbb{R}^d, |\xi|^2 d\xi))'$ given by $\langle g, \cdot \rangle_{L^2(\mathbb{R}^d, d\xi)} \to \langle |\xi|^{-1} g, \cdot \rangle_{L^2(\mathbb{R}^d, |\xi|^2 d\xi)}$ where the latter map is an isomorphism since it closes the diagram

$$\begin{array}{ccc} L^2(\mathbb{R}^d, d\xi) & \overset{f \to |\xi|^{-1} f}{\to} & L^2(\mathbb{R}^d, |\xi|^2 d\xi) \\ f \to \langle f, \cdot \rangle_{L^2(\mathbb{R}^d, d\xi)} \downarrow & & \downarrow f \to \langle f, \cdot \rangle_{L^2(\mathbb{R}^d, |\xi|^2 d\xi)} \\ (L^2(\mathbb{R}^d, d\xi))' & \dashrightarrow & (L^2(\mathbb{R}^d, |\xi|^2 d\xi))' \end{array}$$

Since the other maps in (6.19) are isomorphisms, also the 1st line in (6.19) is an isomorphism. Hence we conclude that $\mathfrak{F} \in (L^2(\mathbb{R}^d, |\xi|^{-2} d\xi))^d$ since $\langle \mathfrak{F}, \cdot \rangle_{L^2(\mathbb{R}^d, d\xi)} \in (L^2(\mathbb{R}^d, |\xi|^2 d\xi))'$ by (6.16).
So we conclude $\mathrm{div}(u \otimes v) \in (\dot{H}^{-1}(\mathbb{R}^d))^d$. Now applying Lemma 2.5 we have in $(\dot{H}^{-1}(\mathbb{R}^d))^d$

$$\mathrm{div}(u \otimes u) = \mathbb{P}\mathrm{div}(u \otimes u) - \nabla p$$

for a function $p \in L^2(\mathbb{R}^d)$ which is what we get in the r.h.s. in (6.1).

We consider now the following truncation of the NS equation.

$$\begin{cases} (u_n)_t + \mathbf{P}_n \mathbb{P}\mathrm{div}(\mathbf{P}_n u_n \otimes \mathbf{P}_n u_n) - \nu(\mathbf{P}_n \triangle) u_n = 0 \\ u_n(0,x) = \mathbf{P}_n u_0(x). \end{cases} \quad (t,x) \in [0,\infty) \times \mathbb{R}^d \tag{6.20}$$

Lemma 6.10. *For any n the system (6.3) admits exactly one solution*

$$u_n \in C^\infty([0,\infty), (H^N(\mathbb{R}^d))^d) \text{ for any } N \in \mathbb{N} \cup \{0\}.$$

Furthermore we have $\mathbb{P} u_n = u_n$ and $\mathbf{P}_n u_n = u_n$.

Proof. First of all, we consider for any n local existence. Set

$$F_n(v) := \nu(\mathbf{P}_n\triangle)v - \mathbf{P}_n\mathbb{P}\mathrm{div}(\mathbf{P}_n v \otimes \mathbf{P}_n v).$$

Then we have

$$\|F_n(v)\|_{(H^N(\mathbb{R}^d))^d} \le \|\nu(\mathbf{P}_n\triangle)v\|_{(H^N(\mathbb{R}^d))^d} + \|\mathbf{P}_n\mathbb{P}\mathrm{div}(\mathbf{P}_n v \otimes \mathbf{P}_n v)\|_{(H^N(\mathbb{R}^d))^d}$$

with

$$\|\nu(\mathbf{P}_n\triangle)v\|_{(H^N(\mathbb{R}^d))^d} \le \nu n^{2+N}\|v\|_{(L^2(\mathbb{R}^d))^d}$$

and

$$\begin{aligned}\|\mathbf{P}_n\mathbb{P}\mathrm{div}(\mathbf{P}_n v \otimes \mathbf{P}_n v)\|_{(H^N(\mathbb{R}^d))^d} &\lesssim n^{N+1}\|\mathbf{P}_n v \otimes \mathbf{P}_n v\|_{L^2} \lesssim n^{N+1}\|\mathbf{P}_n v\|_{L^4}^2 \\ &\lesssim n^{N+1}\|\nabla\mathbf{P}_n v\|_{L^2}^{\frac{d}{4}}\|\nabla\mathbf{P}_n v\|_{L^2}^{\frac{d}{4}}\|\mathbf{P}_n v\|_{L^2}^{1-\frac{d}{4}}\|\mathbf{P}_n v\|_{L^2}^{1-\frac{d}{4}} \\ &\lesssim n^{N+1+\frac{d}{2}}\|v\|_{L^2}^2 \quad .\end{aligned}$$

So for some constant $C_{n,N}$ we have

$$\|F_n(v)\|_{(H^N(\mathbb{R}^d))^d} \le C_{n,N}(\|v\|_{(L^2(\mathbb{R}^d))^d} + \|v\|_{(L^2(\mathbb{R}^d))^d}^2).$$

Furthermore, as a sum of a bounded linear operator and a bounded quadratic form each F_n is a locally Lipchitz function. Then for any n and N we know that (6.3) admits a solution $u_n \in C^1([0, T_{n,N}), (H^N(\mathbb{R}^d))^d))$ for some maximal $T_{n,N} > 0$. Furthermore we must have

$$\lim_{t\nearrow T_{n,N}} \|u_n(t)\|_{(H^N(\mathbb{R}^d))^d} = +\infty \text{ if } T_{n,N} < \infty. \tag{6.21}$$

Next we have $u_n = \mathbb{P}u_n$ since applying $1 - \mathbb{P}$ to (6.20)

$$\begin{cases}((1-\mathbb{P})u_n)_t - \nu(\mathbf{P}_n\triangle)(1-\mathbb{P})u_n = 0 \\ (1-\mathbb{P})u_n(0,x) = 0\end{cases} \Rightarrow (1-\mathbb{P})u_n = 0,$$

and $u_n = \mathbf{P}_n u_n$ since applying $1 - \mathbf{P}_n$ to (6.20)

$$\begin{cases}((1-\mathbf{P}_n)u_n)_t = 0 \\ (1-\mathbf{P}_n)u_n(0,x) = 0\end{cases} \Rightarrow (1-\mathbf{P}_n)u_n = 0$$

Now we show that the *finite time blow up* in (6.21) cannot occur for any (n, N) (in fact, the following argument proves that also *infinite time blow up*, that is (6.21) but with $T_{n,N} = \infty$, cannot occur).
Let us consider (6.21) first in the case $N = 0$. When we apply $\langle \cdot, u_n\rangle_{L^2}$ to the 1st line in (6.3) and get

$$\frac{1}{2}\frac{d}{dt}\|u_n\|_{L^2}^2 + \langle \mathbf{P}_n\mathbb{P}\mathrm{div}(u_n \otimes u_n), u_n\rangle_{L^2} - \nu\langle \triangle u_n, u_n\rangle_{L^2} = 0\,.$$

Notice that summing on repeated indexes $\langle \triangle u_n, \varphi\rangle_{L^2} = -\langle \partial_j u_n, \partial_j \varphi\rangle_{L^2}$ for all $\varphi \in (C_0^\infty(\mathbb{R}^d))^d$ and since this is dense in $(H^1(\mathbb{R}^d))^d$ and both sides define bounded functionals in $(H^1(\mathbb{R}^d))^d$, we conclude

$$\nu\langle \triangle u_n, u_n\rangle_{L^2} = -\nu\|\nabla u_n\|^2_{L^2}.$$

Next, using $\mathbb{P}^* = \mathbb{P}$, $\mathbf{P}_n^* = \mathbf{P}_n$ and (6.17), we have

$$\langle \mathbf{P}_n \mathbb{P}\text{div}(u_n \otimes u_n), u_n\rangle_{L^2} = \langle \text{div}(u_n \otimes u_n), u_n\rangle_{L^2} = 0.$$

Hence we conclude

$$\frac{1}{2}\frac{d}{dt}\|u_n\|^2_{(L^2(\mathbb{R}^d))^d} + \nu\|\nabla u_n\|^2_{(L^2(\mathbb{R}^d))^{d^2}} = 0$$

and we obtain

$$\|u_n(t)\|^2_{(L^2(\mathbb{R}^d))^d} + 2\nu\int_0^t \|\nabla u_n(t')\|^2_{(L^2(\mathbb{R}^d))^{d^2}} dt' = \|\mathbf{P}_n u_0\|^2_{(L^2(\mathbb{R}^d))^d}. \tag{6.22}$$

In particular this yields the bound $\|u_n(t)\|_{L^2} \le \|\mathbf{P}_n u_0\|_{L^2}$ for all $t \in [0, T_{n,0})$ and by (6.21) we conclude that the lifespan is $T_{n,0} = \infty$ for all $n \in \mathbb{N}$. This proves the case $N = 0$ in Lemma 6.10.

Consider now the case $N \in \mathbb{N}$. If $u_n \in C^1([0, T_{n,N}), (H^N(\mathbb{R}^d))^d)$ with $T_{n,N} < \infty$ is a maximal solution, obviously it is the restriction in $[0, T_{n,N})$ of a solution $u_n \in C^1([0,\infty), (L^2(\mathbb{R}^d))^d)$. On the other hand, the blow up (6.21) is impossible because otherwise we would have

$$\infty = \lim_{t\nearrow T_{n,N}} \|u_n(t)\|_{(H^N(\mathbb{R}^d))^d} \le n^N \lim_{t\nearrow T_{n,N}} \|u_n(t)\|_{(L^2(\mathbb{R}^d))^d} \le n^N \|\mathbf{P}_n u_0\|_{L^2} < \infty$$

which is absurd. Hence the lifespan is $T_{n,N} = \infty$ for all $n \in \mathbb{N}$ and $N \in \mathbb{N} \cup \{0\}$.

□

6.1.1 Compactness properties of $\{u_n\}_{n\in\mathbb{N}}$

Now we consider the sequence of solutions $\{u_n\}_{n\in\mathbb{N}}$ of solutions of (6.3). We will prove the following result.

Proposition 6.11. *There exists a $u \in L^\infty(\mathbb{R}_+, (L^2(\mathbb{R}^d))^d) \cap L^2_{loc}(\mathbb{R}_+, H^1(\mathbb{R}^d))^d)$ with $div\, u = 0$ and a subsequence of $\{u_n\}_{n\in\mathbb{N}}$ such that for any $T > 0$ and any compact subset $K \subset \mathbb{R}^d$ we have (after extracting this subsequence)*

$$\lim_{n\to\infty} \int_{[0,T]\times K} |u_n(t,x) - u(t,x)|^2 dtdx = 0. \tag{6.23}$$

Moreover, for all vector fields $\Psi \in L^2([0,T], (H^1(\mathbb{R}^d))^d)$ and all $\Phi \in L^2([0,T]\times\mathbb{R}^d, \mathbb{R}^d)$ we have

$$\lim_{n\to\infty} \int_{[0,T]\times\mathbb{R}^d} (u_n(t,x) - u(t,x)) \cdot \Phi(t,x) dtdx = 0, \tag{6.24}$$

$$\lim_{n\to\infty} \sum_{j,k=1}^d \int_{[0,T]\times\mathbb{R}^d} \partial_k(u_n^j(t,x) - u^j(t,x))\partial_k \Psi^j(t,x) dtdx = 0. \tag{6.25}$$

Finally, for any $\psi \in C^0([0,\infty),(H^1(\mathbb{R}^d))^d)$ we have $\langle u_n,\psi\rangle_{(L^2(\mathbb{R}^d))^d} \to \langle u,\psi\rangle_{(L^2(\mathbb{R}^d))^d}$ in $L^\infty_{loc}([0,\infty))$, that is

$$\lim_{n\to\infty} \|\langle u_n(t)-u(t),\psi(t)\rangle\|_{L^\infty([0,T])} = 0 \text{ for any } T. \tag{6.26}$$

Proof. Fix an arbitrary $T>0$ and an arbitrary compact subset K of $\mathbb{R}^d$.

Claim 6.12. The set $\{u_n\}_{n\in\mathbb{N}}$ gives a relatively compact set in $L^2([0,T]\times K,\mathbb{R}^d)$.

Proof of Claim 6.12. Notice that (6.22) implies that $u_n \in L^2([0,T]\times\mathbb{R}^d,\mathbb{R}^d)$ for all n. We will show the following statement, which is equivalent to Claim 6.12.

Claim 6.13. For any $\varepsilon>0$ there exists a finite family of balls of the space $L^2([0,T]\times K,\mathbb{R}^d)$ which have radius ε and whose union covers the set $\{u_n\}_{n\in\mathbb{N}}$.

Proof of Claim 6.13. First of all, if we want to approximate $\{u_n\}_{n\in\mathbb{N}}$ with $\{\mathbf{P}_{n_0}u_n\}_{n\in\mathbb{N}}$ for a fixed n_0, we can use the fact that for any n_0 and any n we have

$$\|u_n-\mathbf{P}_{n_0}u_n\|_{L^2([0,T]\times\mathbb{R}^d,\mathbb{R}^d)} = \int_0^T \|u_n-\mathbf{P}_{n_0}u_n\|^2_{(L^2(\mathbb{R}^d))^d}dt$$
$$\le n_0^{-2}\int_0^T \|\nabla u_n-\nabla\mathbf{P}_{n_0}u_n\|^2_{(L^2(\mathbb{R}^d))^{d^2}}dt \le 4n_0^{-2}\int_0^T \|\nabla u_n\|^2_{(L^2(\mathbb{R}^d))^{d^2}}dt \le 4n_0^{-2}\|u_0\|^2_{(L^2(\mathbb{R}^d))^d}.$$

Hence we can choose n_0 large enough s.t.

$$\|u_n-\mathbf{P}_{n_0}u_n\|_{L^2([0,T]\times\mathbb{R}^d,\mathbb{R}^d)} < \frac{\varepsilon}{2} \text{ for all } n\in\mathbb{N}. \tag{6.27}$$

Now consider $\{\mathbf{P}_{n_0}u_n\}_{n\in\mathbb{N}}$. Then Claim 6.13 is a consequence of

Claim 6.14. $\{\mathbf{P}_{n_0}u_n\}_{n\in\mathbb{N}}$ is relatively compact in $L^2([0,T]\times K,\mathbb{R}^d)$.

Indeed Claim 6.14 implies that for any $\varepsilon>0$ there is a finite number of balls $B_{L^2([0,T]\times K,\mathbb{R}^d)}(f_j,\frac{\varepsilon}{2})$ which cover $\{\mathbf{P}_{n_0}u_n\}_{n\in\mathbb{N}}$. Hence by (6.27) we conclude that for any $\varepsilon>0$ the balls $B_{L^2([0,T]\times K,\mathbb{R}^d)}(f_j,\varepsilon)$ cover $\{u_n\}_{n\in\mathbb{N}}$ and so we get Claim 6.13.

Proof of Claim 6.14. It will be a consequence of the following stronger claim.

Claim 6.15. $\{\mathbf{P}_{n_0}u_n\}_{n\in\mathbb{N}}$ is relatively compact in $C^0([0,T],(L^2(K))^d)\subset L^\infty([0,T],(L^2(K))^d)$.

Proof of Claim 6.15. To get this result we want to apply the Ascoli–Arzela Theorem (for which a sufficient condition for a sequence of continuous functions $f_n: K\to X$, with K compact and separable metric space and X a complete metric space, to admit a subsequence that converges uniformly to a continuous function $f: K\to X$ is that it is equicontinuous and $\{f_n(k)\}_n$ is relatively compact for any $k\in K$ [1]). So it is enough to show that $\{\mathbf{P}_{n_0}u_n\}_{n\in\mathbb{N}}$

[1] The proof goes as follows. One first considers a dense countable subset $\mathcal{N}$ of K. Then by a diagonal argument, one considers a subsequence $\{f_{n_m}\}$ s.t. $\{f_{n_m}(k)\}$ converges for any $k\in\mathcal{N}$ to a limit that we denote by $f(k)$. Using equicontinuity and the completeness of X it is easy to see that $\{f_{n_m}(k)\}$ converges for any $k\in K$. We denote again by $f(k)$ the limit. Finally, using equicontinuity we conclude that $f: K\to X$ is continuous

is a sequence of equicontinuous functions in $C^0([0,T],(L^2(K))^d)$ and that for any $t\in[0,T]$ the sequence $\{\mathbf{P}_{n_0}u_n(t)\}_{n\in\mathbb{N}}$ is relatively compact in $(L^2(K))^d$.
First of all we want to show that $\{\mathbf{P}_{n_0}u_n\}_{n\in\mathbb{N}}$ is a sequence of equicontinuous functions in $C^0([0,T],(L^2(K))^d)$. This will follow from Hölder (since $\frac{4}{d}>1$ if $d=2,3$) and from the following claim.

Claim 6.16. There exists a fixed constant C s.t.

$$\|(\mathbf{P}_{n_0}u_n)_t\|_{L^{\frac{4}{d}}([0,T],(L^2(\mathbb{R}^d))^d)}\le C \text{ for all } n.$$

Proof of Claim 6.16. We apply $\mathbf{P}_{n_0}$ to (6.3) and we obtain

$$(\mathbf{P}_{n_0}u_n)_t=-\mathbf{P}_{n_0}\mathbf{P}_n\mathbb{P}\mathrm{div}(u_n\otimes u_n)+\nu\mathbf{P}_{n_0}\triangle u_n.$$

We have

$$\|\nu\mathbf{P}_{n_0}\triangle u_n\|_{(L^2(\mathbb{R}^d))^d}\le \nu n_0^2\|u_n\|_{(L^2(\mathbb{R}^d))^d}\le \nu n_0^2\|u_0\|_{(L^2(\mathbb{R}^d))^d}$$

and, by the Gagliardo-Nirenberg inequality,

$$\begin{aligned}&\|\mathbf{P}_{n_0}\mathbf{P}_n\mathbb{P}\mathrm{div}(u_n\otimes u_n)\|_{(L^2(\mathbb{R}^d))^d}\le\|\mathbf{P}_{n_0}\mathrm{div}(u_n\otimes u_n)\|_{(L^2(\mathbb{R}^d))^d}\\ &=\sum_{j=1}^d\|\mathbf{P}_{n_0}\sum_{k=1}^d\partial_k(u_n^ku_n^j)\|_{L^2(\mathbb{R}^d)}\le n_0\sum_{j,k=1}^d\|u_n^ku_n^j\|_{L^2(\mathbb{R}^d)}\\ &\le Cn_0\|u_n\|^2_{(L^4(\mathbb{R}^d))^d}\le C'n_0\left(\|\nabla u_n\|_{L^2}^{\frac{d}{4}}\|u_n\|_{L^2}^{1-\frac{d}{4}}\right)^2.\end{aligned}$$

Then we have

$$\begin{aligned}\|(\mathbf{P}_{n_0}u_n)_t\|_{L^{\frac{4}{d}}([0,T],(L^2(\mathbb{R}^d))^d)}&\le \nu n_0^2T^{\frac{d}{4}}\|u_0\|_{(L^2(\mathbb{R}^d))^d}\\ &\quad+C'n_0\|u_n\|^{2\left(1-\frac{d}{4}\right)}_{L^\infty([0,T],(L^2(\mathbb{R}^d))^d)}\|\nabla u_n\|^{\frac{d}{2}}_{L^2([0,T],L^2}\le C\end{aligned}$$

for some constant C independent of n by the energy equality (6.22) and the fact that $\|\mathbf{P}_nu_0\|_{(L^2(\mathbb{R}^d))^d}\le\|u_0\|_{(L^2(\mathbb{R}^d))^d}$ for all n.
Hence we concluded the proof that $\{\mathbf{P}_{n_0}u_n\}_{n\in\mathbb{N}}$ is a sequence of equicontinuous functions in $C^0([0,T],(L^2(\mathbb{R}^d))^d)$.
To complete the proof of Claim 6.15 we need to show that for any $t\in[0,T]$ the sequence $\{\mathbf{P}_{n_0}u_n(t)\}_{n\in\mathbb{N}}$ is relatively compact in $(L^2(K))^d$. It is here that we will exploit the fact that K is a compact subspace of $\mathbb{R}^d$.
We know that $\{\mathbf{P}_{n_0}u_n(t)\}_{n\in\mathbb{N}}$ is a bounded sequence in $(H^1(\mathbb{R}^d))^d$ for any $t\in[0,T]$. This follows immediately from $\|\mathbf{P}_{n_0}u_n(t)\|_{H^1}\le n_0\|u_n(t)\|_{L^2}\le n_0\|u_0\|_{L^2}$, which follows from the energy inequality (6.22) which guarantees $\|u_n(t)\|_{L^2}\le\|u_0\|_{L^2}$. We recall now that

Claim 6.17. The restriction map $H^1(\mathbb{R}^d)\to L^2(K)$ is compact for any compact K .

Sketch of proof Indeed this is equivalent at showing that

$$\mathcal{T}f := \chi_K \mathcal{F}^* \left(\frac{f}{\langle \xi \rangle} \right) \text{ is compact as } L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d).$$

But we have $\mathcal{T}f = \int \mathcal{K}(x,\xi) f(\xi) d\xi$ with integral *kernel* $\mathcal{K}(x,\xi) := \chi_K(x)\langle \xi \rangle^{-1} e^{-ix\cdot\xi}$. It is easy to see that $\mathcal{T}_n \overset{n\to\infty}{\to} \mathcal{T}$ in the operator norm where the $\mathcal{T}_n$ has kernel $\mathcal{K}_n(x,\xi) := \chi_K(x)\langle \xi \rangle^{-1} e^{-ix\cdot\xi} \chi_{B(0,n)}(\xi)$. Now $\mathcal{K}_n \in L^2(\mathbb{R}^d_x \times \mathbb{R}^d_\xi)$ so that $\mathcal{T}_n$ is Hilbert–Schmidt with $\|\mathcal{T}_n\|_{HS} := \|\mathcal{K}_n\|_{L^2(\mathbb{R}^d_x \times \mathbb{R}^d_\xi)}$. Now it is easy to show that $\|\mathcal{T}_n\|_{L^2\to L^2} \le \|\mathcal{T}_n\|_{HS}$. $\mathcal{K}_n$ is the limit in $L^2(\mathbb{R}^d_x \times \mathbb{R}^d_\xi)$ of elements in $L^2(\mathbb{R}^d_x) \otimes L^2(\mathbb{R}^d_\xi)$. The latter ones are integral kernels of finite rank operators and their operators converge in the Hilbert–Schmidt norm, and so also in the $\|\cdot\|_{L^2\to L^2}$ norm, to $\mathcal{T}_n$. We conclude that there is a sequence of finite rank operators which converges in the operator norm to $\mathcal{T}$, which then is compact. □

It follows that $\{\mathbf{P}_{n_0} u_n(t)\}_{n\in\mathbb{N}}$ is relatively compact in $(L^2(K))^d$ for any $t \in [0,T]$.
Hence the hypotheses of the Ascoli–Arzela Theorem have been checked and we can conclude that Claim 6.15, that is the claim that $\{\mathbf{P}_{n_0} u_n\}_{n\in\mathbb{N}}$ is relatively compact in $C^0([0,T], (L^2(K))^d)$, is true.
Hence there exists a subsequence of $\{u_n\}_{n\in\mathbb{N}}$ (and it is not restrictive to assume this is true for the whole sequence) which converges to an $u \in L^2([0,T] \times K, \mathbb{R}^d)$. By a diagonal argument, we can assume that this is true for any compact $K \subset \mathbb{R}^d$ and any $T > 0$. This yields (6.23). Notice that this implies

$$u_n \to u \text{ in } \mathcal{D}'((0,T) \times \mathbb{R}^d, \mathbb{R}^d). \tag{6.28}$$

We claim now that $u \in L^2([0,T] \times \mathbb{R}^d, \mathbb{R}^d)$ and that

$$u_n \rightharpoonup u \text{ in } L^2([0,T] \times \mathbb{R}^d, \mathbb{R}^d) \tag{6.29}$$

(convergence in the weak topology). Indeed, since from (6.22) we have that $\{u_n\}_{n\in\mathbb{N}}$ is uniformly bounded in $L^2([0,T] \times \mathbb{R}^d, \mathbb{R}^d)$, it follows that up to a subsequence we have $u_n \rightharpoonup v$ for some $v \in L^2([0,T] \times \mathbb{R}^d, \mathbb{R}^d)$. Then (6.28) implies that $v = u$ as distributions in $\mathcal{D}'((0,T) \times \mathbb{R}^d, \mathbb{R}^d)$. This implies that $u \in L^2([0,T] \times \mathbb{R}^d, \mathbb{R}^d)$ with $u = v$.
In particular this implies

$$\lim_{n\to\infty} \int_{[0,T]\times\mathbb{R}^d} (u_n(t,x) - u(t,x)) \cdot \Phi(t,x) dtdx = 0 \text{ for all } \Phi \in L^2([0,T] \times \mathbb{R}^d, \mathbb{R}^d),$$

that is (6.24).

We now turn to the proof of (6.25).
By (6.22) we know that $\{\nabla u_n\}_{n\in\mathbb{N}}$ is bounded in $L^2((0,T)\times\mathbb{R}^d, (\mathbb{R}^d)^d)$. This implies that up to a subsequence there exists $V \in L^2((0,T) \times \mathbb{R}^d, (\mathbb{R}^d)^d)$ s.t. $\nabla u_n \rightharpoonup V$. On the other hand (6.29) implies $u_n \to u$ in $\mathcal{D}'((0,T)\times\mathbb{R}^d)$. This in turn implies $\partial_j u_n \to \partial_j u$ in $\mathcal{D}'((0,T)\times\mathbb{R}^d)$ for any $j = 1, ..., d$. Hence $\nabla u = V$ in $\mathcal{D}'((0,T) \times \mathbb{R}^d)$, $\nabla u \in L^2((0,T) \times \mathbb{R}^d, (\mathbb{R}^d)^d)$ and $\nabla u = V$ in $L^2((0,T) \times \mathbb{R}^d, (\mathbb{R}^d)^d)$. This proves (6.25).

Notice also that, up to a subsequence, $u_n(t,x) \to u(t,x)$ and $\nabla u_n(t,x) \to \nabla u(t,x)$ almost everywhere. Then the energy inequalities (6.22) imply by Fathou

$$\|u(t)\|^2_{L^2(\mathbb{R}^d)} + 2\nu \int_0^t \|\nabla u(t')\|^2_{L^2(\mathbb{R}^d)} dt' \le \|u_0\|^2_{L^2(\mathbb{R}^d)}. \tag{6.6}$$

We turn now to the proof of (6.26).
Fix a function $\psi \in C^0([0,\infty), \mathbb{P}(H^1(\mathbb{R}^d))^d)$ like in the statement of Proposition 6.11. For a given n_0 consider

$$g_n(t) := \langle u_n(t), \psi(t)\rangle_{(L^2(\mathbb{R}^d))^d} \text{ and } g_n^{(n_0)}(t) := \langle \mathbf{P}_{n_0} u_n(t), \psi(t)\rangle_{(L^2(\mathbb{R}^d))^d}.$$

Then for any $\epsilon > 0$ and any fixed $T > 0$ there exists n_0 s.t.

$$\|(\mathbf{P}_{n_0} - 1)\psi(t)\|_{L^\infty([0,T],(L^2(\mathbb{R}^d))^d)} < \epsilon.$$

This and $\|u_n(t)\|_{L^\infty([0,T],(L^2(\mathbb{R}^d))^d)} \le \|u_0\|_{(L^2(\mathbb{R}^d))^d}$ imply

$$\|g_n - g_n^{(n_0)}\|_{L^\infty([0,T]} \le \|u_0\|_{(L^2(\mathbb{R}^d))^d} T\epsilon.$$

Furthermore, for any fixed $T > 0$ there exists a compact K s.t.

$$\|\psi(t)\|_{L^\infty([0,T],(L^2(\mathbb{R}^d \setminus K))^d)} < \epsilon.$$

Then, if we set $g_n^{(n_0,K)}(t) := \langle \mathbf{P}_{n_0} u_n(t), \psi(t)\rangle_{(L^2(K)))^d}$ we have

$$\|g_n^{(n_0,K)} - g_n^{(n_0)}\|_{L^\infty([0,T]} \le \|u_0\|_{(L^2(\mathbb{R}^d))^d} T\epsilon.$$

Since by Claim 6.15 we know that $\mathbf{P}_{n_0} u_n \to \mathbf{P}_{n_0} u$ in $C^0([0,T],(L^2(K))^d)$, we conclude that

$$\{g_n^{(n_0,K)}\}_n = \langle \mathbf{P}_{n_0} u_n(t), \psi(t)\rangle_{(L^2(K)))^d} \to \langle \mathbf{P}_{n_0} u(t), \psi(t)\rangle_{(L^2(K)))^d} \text{ in } C^0([0,T]).$$

But then also

$$\begin{aligned}
&\|\langle u_n(t), \psi(t)\rangle_{(L^2(\mathbb{R}^d))^d} - \langle u(t), \psi(t)\rangle_{(L^2(\mathbb{R}^d))^d}\|_{L^\infty([0,T]} \\
&\le \|\langle \mathbf{P}_{n_0} u_n(t), \psi(t)\rangle_{(L^2(K))^d} - \langle \mathbf{P}_{n_0} u(t), \psi(t)\rangle_{(L^2(K))^d}\|_{L^\infty([0,T]} + 2\|u_0\|_{(L^2(\mathbb{R}^d))^d} T\epsilon \\
&+ \langle u(t), (1 - \mathbf{P}_{n_0})\psi(t)\rangle_{(L^2(\mathbb{R}^d))^d}\|_{L^\infty([0,T]} + \langle u(t), (1 - \chi_K)\psi(t)\rangle_{(L^2(\mathbb{R}^d))^d}\|_{L^\infty([0,T]} \le \\
&\le \|\langle \mathbf{P}_{n_0} u_n(t), \psi(t)\rangle_{(L^2(K))^d} - \langle \mathbf{P}_{n_0} u(t), \psi(t)\rangle_{(L^2(K))^d}\|_{L^\infty([0,T]} + 4\|u_0\|_{(L^2(\mathbb{R}^d))^d} T\epsilon.
\end{aligned}$$

Since T is fixed and ϵ is arbitrarily small, it follows that we obtain that g_n converges to $\langle u(t), \psi(t)\rangle_{(L^2(\mathbb{R}^d))^d}$ in $L^\infty([0,T])$, and hence in $C^0([0,T])$. In particular we have shown that $u \in C^0([0,\infty), L^2_w(\mathbb{R}^d, \mathbb{R}^d))$. The proof of Proposition 6.11 is completed. □

6.1.2 End of the proof of Leray's Theorem 6.2

Proposition 6.11 has provided us with a function

$$u \in L^\infty([0,\infty), L^2(\mathbb{R}^d,\mathbb{R}^d)) \cap L^2_{loc}([0,\infty), H^1(\mathbb{R}^d,\mathbb{R}^d)) \cap C^0([0,\infty), L^2_w(\mathbb{R}^d,\mathbb{R}^d))$$

which satisfies the energy inequality

$$\|u(t)\|^2_{L^2(\mathbb{R}^d)} + 2\nu \int_0^t \|\nabla u(t')\|^2_{L^2(\mathbb{R}^d)} dt' \le \|u_0\|^2_{L^2(\mathbb{R}^d)}. \tag{6.6}$$

Our aim in this section is to prove that u is a weak solution in the sense of Definition 6.1. Let us consider $\Psi \in C^1([0,\infty), \mathbb{P}(H^1(\mathbb{R}^d))^d)$ and let us apply to (6.3) the inner product $\langle \cdot, \Psi\rangle_{L^2}$. Then we get

$$\langle (u_n)_t, \Psi\rangle_{(L^2(\mathbb{R}^d))^d} + \langle \mathbf{P}_n \mathbb{P}\mathrm{div}(u_n \otimes u_n), \Psi\rangle_{(L^2(\mathbb{R}^d))^d} - \nu \langle \triangle u_n, \Psi\rangle_{(L^2(\mathbb{R}^d))^d} = 0.$$

Hence

$$\frac{d}{dt}\langle u_n, \Psi\rangle_{(L^2(\mathbb{R}^d))^d} - \langle u_n, \Psi_t\rangle_{(L^2(\mathbb{R}^d))^d} + \langle \mathrm{div}(u_n \otimes u_n), \mathbf{P}_n\Psi\rangle_{(L^2(\mathbb{R}^d))^d} + \nu \langle \triangle u_n, \Psi\rangle_{(L^2(\mathbb{R}^d))^d} = 0.$$

So, integrating in t we get

$$\begin{aligned}\int_{\mathbb{R}^d} u_n(t,x)\cdot\Psi(t,x)dx &= \int_{\mathbb{R}^d} \mathbf{P}_n u_0(x)\cdot \Psi(0,x)dx - \int_0^t ds \int_{\mathbb{R}^d} u_n(s,x)\otimes u_n(s,x) : \nabla \mathbf{P}_n \Psi(s,x)dx \\ &+ \int_0^t ds \int_{\mathbb{R}^d} u_n(s,x)\cdot \Psi_t(s,x)dx - \nu \sum_{j,k} \int_0^t ds \int_{\mathbb{R}^d} \partial_k u_n^j(s,x)\partial_k \Psi^j(s,x)dx. \end{aligned} \tag{6.30}$$

By (6.26) for any t

$$\lim_{n\to\infty} \int_{\mathbb{R}^d} u_n(t,x)\cdot\Psi(t,x)dx = \int_{\mathbb{R}^d} u(t,x)\cdot\Psi(t,x)dx. \tag{6.31}$$

By the definition of $\mathbf{P}_n$ we have

$$\lim_{n\to\infty} \int_{\mathbb{R}^d} \mathbf{P}_n u_0(x)\cdot\Psi(0,x)dx = \int_{\mathbb{R}^d} u_0(x)\cdot\Psi(0,x)dx. \tag{6.32}$$

By (6.24) we have

$$\lim_{n\to\infty} \int_0^t ds \int_{\mathbb{R}^d} u_n(s,x)\cdot\Psi_t(s,x)dx = \int_0^t ds \int_{\mathbb{R}^d} u(s,x)\cdot\Psi_t(s,x)dx. \tag{6.33}$$

By (6.25) we have

$$\lim_{n\to\infty} \nu \int_0^t ds \int_{\mathbb{R}^d} \partial_k u_n^j(s,x)\partial_k\Psi^j(s,x)dx = \nu \int_0^t ds \int_{\mathbb{R}^d} \partial_k u^j(s,x)\partial_k\Psi^j(s,x)dx. \tag{6.34}$$

The above limits (6.31)–(6.34) are straightforward consequences of Proposition 6.11. By taking the limit in (6.30), Leray's Theorem will be a consequence of the following claim, which is the delicate point of this part of the proof.

Claim 6.18. We have

$$\lim_{n\to\infty}\int_0^t ds\int_{\mathbb{R}^d} u_n(s,x)\otimes u_n(s,x):\nabla \mathbf{P}_n\Psi(s,x)dx=\int_0^t ds\int_{\mathbb{R}^d} u(s,x)\otimes u(s,x):\nabla\Psi(s,x)dx. \tag{6.35}$$

Proof of Claim 6.18. The 1st step, algebraic, is to write

$$\begin{aligned}\int_0^t ds\int_{\mathbb{R}^d} u_n(s,x)\otimes u_n(s,x):\nabla \mathbf{P}_n\Psi(s,x)dx&=\int_0^t ds\int_{\mathbb{R}^d} u_n(s,x)\otimes u_n(s,x):\nabla\Psi(s,x)dx\\&+\int_0^t ds\int_{\mathbb{R}^d} u_n(s,x)\otimes u_n(s,x):\nabla(\mathbf{P}_n\Psi(s,x)-\Psi(s,x))dx.\end{aligned}$$

Claim 6.18 will be a consequence of

$$\lim_{n\to\infty}\int_0^t ds\int_{\mathbb{R}^d} u_n(s,x)\otimes u_n(s,x):\nabla\Psi(s,x)dx=\int_0^t ds\int_{\mathbb{R}^d} u(s,x)\otimes u(s,x):\nabla\Psi(s,x)dx. \tag{6.36}$$

and of

$$\lim_{n\to\infty}\int_0^t ds\int_{\mathbb{R}^d} u_n(s,x)\otimes u_n(s,x):\nabla(\mathbf{P}_n\Psi(s,x)-\Psi(s,x))dx=0. \tag{6.37}$$

In order to prove (6.36)–(6.37) we observe that since $\Psi\in C^1([0,\infty),(H^1(\mathbb{R}^d))^d)$ for any $\varepsilon>0$ there is a compact set $K\subset\mathbb{R}^d$ s.t.

$$\sup_{s\in[0,T]}\|\nabla\Psi(s,\cdot)\|_{L^2(\mathbb{R}^d\backslash K)}<\varepsilon. \tag{6.38}$$

(6.38) is elementary to prove and it is assumed in the sequel. Now we show (6.36).
By Hölder, (6.38), Gagliardo–Nirenberg and the energy equality (6.22) we have

$$\begin{aligned}&|\int_0^t ds\int_{\mathbb{R}^d\backslash K} u_n(s,x)\otimes u_n(s,x):\nabla\Psi(s,x)dx|\le\int_0^T ds\|u_n\otimes u_n\|_{L^2(\mathbb{R}^d)}\|\nabla\Psi(s)\|_{L^2(\mathbb{R}^d\backslash K)}\\&\le T^{\frac{4-d}{4}}\|u_n\otimes u_n\|_{L^{\frac{4}{d}}([0,T],L^2(\mathbb{R}^d))}\|\nabla\Psi\|_{L^\infty([0,T],L^2(\mathbb{R}^d\backslash K))}\\&\le T^{\frac{4-d}{4}}\|\|u_n\|^2_{L^4(\mathbb{R}^d)}\|_{L^{\frac{4}{d}}(0,T)}\varepsilon\lesssim\varepsilon T^{\frac{4-d}{4}}\|\|u_n\|^{2(1-d/4)}_{L^2(\mathbb{R}^d)}\|\nabla u_n\|^{d/2}_{L^2(\mathbb{R}^d)}\|_{L^{\frac{4}{d}}(0,T)}\\&\lesssim\varepsilon T^{\frac{4-d}{4}}\|u_n\|^{2\left(1-\frac{d}{4}\right)}_{L^\infty([0,T],L^2(\mathbb{R}^d))}\|\nabla u_n\|^{\frac{d}{2}}_{L^2([0,T],L^2(\mathbb{R}^d))}\le\varepsilon T^{\frac{4-d}{4}}\|u_0\|^2_{L^2(\mathbb{R}^d)}.\end{aligned}$$

Hence, to prove (6.36) it is enough to show for any compact set $K\subset\mathbb{R}^d$

$$\lim_{n\to\infty}\int_0^t ds\int_K u_n(s,x)\otimes u_n(s,x):\nabla\Psi(s,x)dx=\int_0^t ds\int_K u(s,x)\otimes u(s,x):\nabla\Psi(s,x)dx. \tag{6.39}$$

The limit (6.39) is a consequence of

$$\lim_{n\to\infty} u_n \otimes u_n = u \otimes u \text{ in } L^1([0,T], L^2(K))$$

which in turn is a consequence of

$$\lim_{n\to\infty} u_n = u \text{ in } L^2([0,T], L^4(K)). \tag{6.40}$$

Let us consider $\chi \in C_c^\infty(\mathbb{R}^d, [0,1])$ s.t. $\chi = 1$ in K, $\Omega := \text{supp}\chi$ and with $\|\nabla\chi\|_{L^\infty(\mathbb{R}^d)} \leq 1$. Then by Gagliardo Nirenberg we have

$$\|f\|_{L^4(K)} \leq C\|f\|_{L^2(\Omega)}^{1-d/4}(\|\chi\nabla f\|_{L^2(\mathbb{R}^d)} + \|f\nabla\chi\|_{L^2(\mathbb{R}^d)})^{d/4} \leq C\|f\|_{L^2(\Omega)}^{1-d/4}\|f\|_{H^1(\mathbb{R}^d)}^{d/4}.$$

Using this inequality and Hölder (using $\frac{1}{2} = \frac{4-d}{8} + \frac{d}{8}$):

$$\begin{aligned}
&\|u - u_n\|_{L^2([0,T],L^4(K))} \lesssim \|\|u-u_n\|_{L^2(\Omega)}^{1-\frac{d}{4}}\|u-u_n\|_{H^1(\mathbb{R}^d)}^{\frac{d}{4}}\|_{L^2(0,T)} \\
&\leq \|\|u-u_n\|_{L^2(\Omega)}^{1-\frac{d}{4}}\|_{L^{\frac{8}{4-d}}(0,T)}\|\|u-u_n\|_{H^1(\mathbb{R}^d)}^{\frac{d}{4}}\|_{L^{\frac{8}{d}}(0,T)} \\
&= \|u-u_n\|_{L^2([0,T],L^2(\Omega))}^{1-\frac{d}{4}}\|u-u_n\|_{L^2([0,T],H^1(\mathbb{R}^d))}^{\frac{d}{4}} \\
&\leq (2(1+\sqrt{T})\|u_0\|_{(L^2(\mathbb{R}^d))^d})^{\frac{d}{4}}\|u-u_n\|_{L^2([0,T],L^2(\Omega))}^{1-\frac{d}{4}} \overset{n\to\infty}{\to} 0
\end{aligned}$$

where the limit holds because $u_n \overset{n\to\infty}{\to} u$ in $L^\infty([0,T], (L^2(\Omega))^d)$, as we saw earlier. This yields (6.40) and so also (6.39).
The proof of (6.37) will follow from the fact that for any $\varepsilon > 0$ there is N s.t. $n \geq N$ implies

$$\sup_{s\in[0,T]} \|\nabla(\mathbf{P}_n\Psi(s) - \Psi(s))\|_{L^2(\mathbb{R}^d)} < \varepsilon$$

In turn this, like (6.38), is a simple consequence of the fact that $\Psi \in C^1([0,\infty), (H^1(\mathbb{R}^d))^d)$. To prove (6.37) observe that

$$\begin{aligned}
&|\text{r.h.s. of (6.37)}| \leq \|u_n \otimes u_n\|_{L^1([0,T],(L^2(\mathbb{R}^d))^{d^2})}\|\nabla(\mathbf{P}_n\Psi - \Psi)\|_{L^2([0,T],(L^2(\mathbb{R}^d))^d)} \\
&\leq \varepsilon\|u_n\|_{L^2([0,T],(L^4(\mathbb{R}^d))^d)}^2 = \varepsilon\|\|u_n\|_{L^4(\mathbb{R}^d)}\|_{L^2(0,T)}^2 \lesssim \varepsilon\|\|u_n\|_{L^2(\mathbb{R}^d)}^{1-d/4}\|\nabla u_n\|_{L^2(\mathbb{R}^d)}^{d/4}\|_{L^2(0,T)}^2 \\
&\lesssim \varepsilon\|u_n\|_{L^\infty([0,T],L^2(\mathbb{R}^d))}^{2(1-\frac{d}{4})}\|\nabla u_n\|_{L^2([0,T],L^2(\mathbb{R}^d))}^{\frac{d}{2}} \leq T^{1-\frac{d}{4}}\varepsilon\|u_0\|_{(L^2(\mathbb{R}^d))^d}^2 \overset{\varepsilon\to 0}{\to} 0
\end{aligned}$$

This completes the proof of Leray's Theorem 6.2.

□

In the next section we prove the 2nd Leray's theorem, that is Theorem 6.3. Uniqueness in this case will follow from the fact that we will frame the problem as a fixed point argument using a contraction.

7 Well posedness in Sobolev spaces

For this section see [1].

Consider the equation (6.7). If $\mathcal{Q}_{NS}(u,u)$ is a force like the f in (5.1), we can interpret the solutions of (6.7) as solutions of a linear heat equation (5.1). We denote by $B(u,v)$ the weak solution of

$$\begin{cases} \partial_t B(u,v) - \nu \triangle B(u,v) = \mathcal{Q}_{NS}(u,v) \\ \qquad B(u,v)|_{t=0} = 0. \end{cases} \tag{7.1}$$

Then, when we are within the scope of the theory of Sect. 5, the solutions of (6.7) can be rewritten as

$$u = e^{\nu t \triangle} u_0 + B(u,u). \tag{7.2}$$

In the sequel we will use repeatedly the following abstract lemma.

Lemma 7.1. *Let X be a Banach space and $B : X^2 \to X$ a continuous bilinear map. Let $\alpha < \frac{1}{4\|B\|}$ where $\|B\| = \sup_{\|x\|=\|y\|=1} \|B(x,y)\|$. Then for any $x_0 \in X$ in $D_X(0,\alpha)$ (the open ball of center 0 and radius α in X) there exists a unique $x \in \overline{D}_X(0,2\alpha)$ s.t. $x = x_0 + B(x,x)$.*

Proof. We consider the map

$$x \to x_0 + B(x,x). \tag{7.3}$$

We will frame this as a fixed point problem in $\overline{D}_X(0,2\alpha)$.
First of all, we claim that the map (7.3) leaves $\overline{D}_X(0,2\alpha)$ invariant. Indeed

$$\|x_0 + B(x,x)\| \le \|x_0\| + \|B(x,x)\| \le \|x_0\| + \|B\|\|x\|^2 \le \alpha \overbrace{(1 + \underbrace{4\|B\|\alpha}_{<1})}^{\le 2} < 2\alpha.$$

Next, we check that the map (7.3) is a contraction. Indeed

$$\|B(x,x) - B(y,y)\| \le \|B(x-y,x)\| + \|B(y,x-y)\| \le 4\alpha\|B\|\|x-y\|$$

where $4\alpha\|B\| < 1$. So the map (7.3) has a unique fixed point in $\overline{D}_X(0,2\alpha)$.

□

Using the above lemma we will prove the following well posedness result.

Theorem 7.2. *For any $u_0 \in (\dot{H}^{\frac{d}{2}-1}(\mathbb{R}^d))^d$ there exists a T and a solution of (7.2) with $u \in L^4([0,T], (\dot{H}^{\frac{d-1}{2}}(\mathbb{R}^d))^d)$. This solution is unique. Furthermore we have*

$$u \in C([0,T], (\dot{H}^{\frac{d}{2}-1}(\mathbb{R}^d))^d), \ \nabla u \in L^2([0,T], (\dot{H}^{\frac{d}{2}-1}(\mathbb{R}^d))^{d^2}). \tag{7.4}$$

Let T_{u_0} be the lifespan of the solution. Then:

(1) there exists a c s.t.

$$\|u_0\|_{(\dot{H}^{\frac{d}{2}-1}(\mathbb{R}^d))^d} \le c\nu \Rightarrow T_{u_0} = \infty;$$

(2) if $T_{u_0} < \infty$ then

$$\int_0^{T_{u_0}} \|u(t)\|^4_{(\dot H^{\frac{d-1}{2}}(\mathbb{R}^d))^d} dt = \infty. \tag{7.5}$$

(3) if $T_{u_0} < \infty$ then

$$\int_0^{T_{u_0}} \|\nabla u(t)\|^2_{\dot H^{\frac{d}{2}-1}} dt = \infty. \tag{7.6}$$

Moreover, if u and v are solutions, then

$$\begin{aligned} &\|u(t)-v(t)\|^2_{(\dot H^{\frac{d}{2}-1}(\mathbb{R}^d))^d} + \nu \int_0^t \|\nabla(u-v)(s)\|^2_{(\dot H^{\frac{d}{2}-1}(\mathbb{R}^d))^d} ds \\ &\le \|u_0-v_0\|^2_{(\dot H^{\frac{d}{2}-1}(\mathbb{R}^d))^d} e^{C\nu^{-3}\int_0^t \left(\|u(t')\|^4_{(\dot H^{\frac{d-1}{2}}(\mathbb{R}^d))^d} + \|v(t')\|^4_{(\dot H^{\frac{d-1}{2}}(\mathbb{R}^d))^d}\right) dt'} \end{aligned} \tag{7.7}$$

where C is a fixed constant.

Remark 7.3. While for $d=2$ the solutions provided by Theorem 7.2 are exactly Leray's solutions, for $d=3$ we could have $u_0 \in (\dot H^{\frac{1}{2}}(\mathbb{R}^3))^3$ with $u_0 \notin (L^2(\mathbb{R}^3))^3$. The corresponding solutions of the Navier Stokes equations provided by Theorem 7.2 are not Leray's solutions.

Remark 7.4. Notice that the finite lifespan (7.5) is relevant only for $d=3$. Furthermore, if $T_{u_0} < \infty$, it has been shown that

$$\|u\|_{L^\infty([0,T_{u_0}],(\dot H^1(\mathbb{R}^3))^3)} = \infty,$$

but the proof is a much harder.
There is no blow up at $T=\infty$. Indeed, we will see in Sect. 8.1 that if $T_{u_0}=\infty$ we have $\lim_{t\to+\infty} \|u(t)\|_{\dot H^{\frac{1}{2}}(\mathbb{R}^3)} = 0$.

We will assume for the moment Theorem 7.2 and prove the following.

8 Proof of Theorem 7.2

This section is devoted to the proof of this theorem. First we have the following lemma.

Lemma 8.1. *Let $d=2,3$. There exists a constant $C>0$ s.t.*

$$\|\mathcal{Q}_{NS}(u,v)\|_{\dot H^{\frac{d}{2}-2}(\mathbb{R}^d,\mathbb{R}^d)} \le C\|u\|_{\dot H^{\frac{d-1}{2}}(\mathbb{R}^d,\mathbb{R}^d)} \|v\|_{\dot H^{\frac{d-1}{2}}(\mathbb{R}^d,\mathbb{R}^d)}. \tag{8.1}$$

Proof. If $d=2$ we have

$$\begin{aligned} \|\mathcal{Q}_{NS}(u,v)\|_{\dot H^{-1}} &\le \sum_{j,k=1}^2 \left(\|\partial_k(u^k v^j)\|_{\dot H^{-1}} + \|\partial_k(v^k u^j)\|_{\dot H^{-1}}\right) \\ &\le 2\sum_{j,k}^2 \|u^k v^j\|_{L^2} \le C\|u\|_{L^4}\|v\|_{L^4} \le C\|u\|_{\dot H^{\frac{1}{2}}}\|v\|_{\dot H^{\frac{1}{2}}} \end{aligned}$$

by the Sobolev embedding $\dot H^{\frac12}(\mathbb{R}^2)\subset L^4(\mathbb{R}^2)$, since $\frac14=\frac12-\frac{\frac12}{2}$. This yields (8.1) for $d=2$.
For $d=3$

$$\begin{aligned}\|\mathcal{Q}_{NS}(u,v)\|_{\dot H^{-\frac12}(\mathbb{R}^3)} &\le \sum_{j,k}^{2}\left(\|\partial_k(u^kv^j)\|_{\dot H^{-\frac12}(\mathbb{R}^3)}+\|\partial_k(v^ku^j)\|_{\dot H^{-\frac12}(\mathbb{R}^3)}\right)\\ &\lesssim \|(\nabla u)v\|_{\dot H^{-\frac12}(\mathbb{R}^3)}+\|u\nabla v\|_{\dot H^{-\frac12}(\mathbb{R}^3)}\lesssim \|(\nabla u)v\|_{L^{\frac32}(\mathbb{R}^3)}+\|u\nabla v\|_{L^{\frac32}(\mathbb{R}^3)}\end{aligned}$$

where we are using the Sobolev embedding $\dot H^{\frac12}(\mathbb{R}^3)\subset L^3(\mathbb{R}^3)$ (since $\frac13=\frac12-\frac{\frac12}{3}$) which in turn by duality implies $L^{\frac32}(\mathbb{R}^3)\subset \dot H^{-\frac12}(\mathbb{R}^3)$.
Hence, by $\frac23=\frac12+\frac16$ and Hölder,

$$\|\mathcal{Q}_{NS}(u,v)\|_{\dot H^{-\frac12}(\mathbb{R}^3)}\lesssim \|\nabla u\|_{L^2(\mathbb{R}^3)}\|v\|_{L^6(\mathbb{R}^3)}+\|u\|_{L^6(\mathbb{R}^3)}\|\nabla v\|_{L^2(\mathbb{R}^3)}\le 2\|u\|_{\dot H^1(\mathbb{R}^3)}\|v\|_{\dot H^1(\mathbb{R}^3)}.$$

This yields (8.1) for $d=3$. □

A straightforward consequence of Lemma 8.1 is the following for C the constant in Lemma 8.1.

Lemma 8.2. *Let $d=2,3$. Then for $u,v\in L^4([0,T],(\dot H^{\frac{d-1}{2}}(\mathbb{R}^d,\mathbb{R}^d))$ we have*

$$\|\mathcal{Q}_{NS}(u,v)\|_{L^2([0,T],\dot H^{\frac d2-2}(\mathbb{R}^d,\mathbb{R}^d))}\le C\|u\|_{L^4([0,T],\dot H^{\frac{d-1}{2}}(\mathbb{R}^d,\mathbb{R}^d))}\|v\|_{L^4([0,T],\dot H^{\frac{d-1}{2}}(\mathbb{R}^d,\mathbb{R}^d))} \tag{8.2}$$

□

Proof of Theorem 7.2. By Theorem 5.4 we have for $s=\frac d2-1$ and $p=4$

$$\begin{aligned}\|B(u,v)\|_{L^4([0,T],\dot H^{\frac{d-1}{2}})} &= \|\|B(u,v)\|_{\dot H^{s+\frac24}}\|_{L^p(0,T)}\lesssim \frac{1}{\nu^{\frac1p+\frac12}}\|\mathcal{Q}_{NS}(u,v)\|_{L^2([0,T],\dot H^{s-1})}\\ &=\nu^{-\frac34}\|\mathcal{Q}_{NS}(u,v)\|_{L^2([0,T],\dot H^{\frac d2-2})}\le C\nu^{-\frac34}\|u\|_{L^4([0,T],\dot H^{\frac{d-1}{2}}}\|v\|_{L^4([0,T],\dot H^{\frac{d-1}{2}})}.\end{aligned} \tag{8.3}$$

So in the Banach space $X=L^4([0,T],\dot H^{\frac{d-1}{2}})$ we have $\|B\|\le C\nu^{-\frac34}$. Obviously this is the same as $\frac{\nu^{\frac34}}{4C}\le\frac{1}{4\|B\|}$. Our strategy is to prove

$$\|e^{\nu t\triangle}u_0\|_{L^4([0,T],\dot H^{\frac{d-1}{2}})}<\frac{\nu^{\frac34}}{4C}\le\frac{1}{4\|B\|} \tag{8.4}$$

where $e^{\nu t\triangle}u_0$ plays the role of x_0 in the abstract Lemma 7.1.
If (8.4) happens, that is if the l.h.s.of (8.4) is less than an $\alpha<\frac{1}{4\|B\|}$, then by Lemma 7.1 we can conclude that problem (7.2) admits a unique solution in $L^4([0,T],\dot H^{\frac{d-1}{2}})$ with norm less than $2\alpha<\frac{\nu^{\frac34}}{2C}$.
We consider two distinct proofs of (8.4). The 1st, simpler, is valid only if $\|u_0\|_{\dot H^{\frac d2-1}}$ is sufficiently small and shows that (8.4) holds for all T. In the second proof, which is general, we drop the assumption that $\|u_0\|_{\dot H^{\frac d2-1}}$ is small, and we prove (8.4) for T sufficiently small.

Step 1: small initial data. By Theorem 5.4 we have for $s = \frac{d}{2} - 1$ and $p = 4$

$$\|e^{\nu t\triangle}u_0\|_{L^4([0,T],\dot{H}^{\frac{d-1}{2}})} = \|\|e^{\nu t\triangle}u_0\|_{\dot{H}^{s+\frac{2}{p}}}\|_{L^p(0,T)} \le \nu^{-\frac{1}{p}}\|u_0\|_{\dot{H}^s} = \nu^{-\frac{1}{4}}\|u_0\|_{\dot{H}^{\frac{d}{2}-1}}. \tag{8.5}$$

So, if $\|u_0\|_{\dot{H}^{\frac{d}{2}-1}} < \frac{\nu}{4C}$ then (8.4) is true for any $T > 0$. In particular $T_{u_0} = \infty$ and we have just proved (1) in Theorem 7.2.

Step 2: possibly large initial data. Now we consider the case when $u_0 \in \dot{H}^{\frac{d}{2}-1}(\mathbb{R}^d)$ is possibly large. We consider a low–high energy decomposition: $u_0 = \mathbf{P}_\rho u_0 + \chi_{\sqrt{-\triangle}\ge\rho}u_0$ where we pick $\rho = \rho_{u_0}$ large enough so that

$$\|\chi_{\sqrt{-\triangle}\ge\rho}u_0\|_{\dot{H}^{\frac{d}{2}-1}} < \frac{\nu}{8C}.$$

Then by (8.5) we get

$$\begin{aligned}\|e^{\nu t\triangle}u_0\|_{L^4([0,T],\dot{H}^{\frac{d-1}{2}})} &\le \|e^{\nu t\triangle}\chi_{\sqrt{-\triangle}\ge\rho}u_0\|_{L^4([0,T],\dot{H}^{\frac{d-1}{2}})} + \|e^{\nu t\triangle}\mathbf{P}_\rho u_0\|_{L^4([0,T],\dot{H}^{\frac{d-1}{2}})}\\ &< \frac{\nu^{\frac{3}{4}}}{8C} + \|e^{\nu t\triangle}\mathbf{P}_\rho u_0\|_{L^4([0,T],\dot{H}^{\frac{d-1}{2}})}\end{aligned} \tag{8.6}$$

where we made the high energy contribution small by the choice of ρ large.
We now exploit the fact that we have the freedom to choose T small, in order to make the contribution to (8.6) small too. Indeed we have

$$\begin{aligned}&\|e^{\nu t\triangle}\mathbf{P}_\rho u_0\|_{L^4([0,T],\dot{H}^{\frac{d-1}{2}})} = \|e^{\nu t\triangle}\chi_{[0,\rho]}(\sqrt{-\triangle})u_0\|_{L^4([0,T],\dot{H}^{\frac{d-1}{2}})}\\ &= \|e^{\nu t\triangle}\chi_{[0,\rho]}(\sqrt{-\triangle})\sqrt{\rho}\frac{(-\triangle)^{\frac{1}{4}}}{\sqrt{\rho}}u_0\|_{L^4([0,T],\dot{H}^{\frac{d}{2}-1})}\\ &\le \sqrt{\rho}\|e^{\nu t\triangle}\chi_{[0,\rho]}(\sqrt{-\triangle})u_0\|_{L^4([0,T],\dot{H}^{\frac{d}{2}-1})} = \sqrt{\rho}\|e^{\nu t\triangle}\mathbf{P}_\rho u_0\|_{L^4([0,T],\dot{H}^{\frac{d}{2}-1})}\\ &\le (\rho^2 T)^{\frac{1}{4}}\|e^{\nu t\triangle}\mathbf{P}_\rho u_0\|_{L^\infty([0,T],\dot{H}^{\frac{d}{2}-1})} \le (\rho^2 T)^{\frac{1}{4}}\|\mathbf{P}_\rho u_0\|_{\dot{H}^{\frac{d}{2}-1}} \le (\rho^2 T)^{\frac{1}{4}}\|u_0\|_{\dot{H}^{\frac{d}{2}-1}} \le \frac{\nu^{\frac{3}{4}}}{8C}\end{aligned}$$

if we choose T small enough so that the last inequality holds, that is if we choose T such that

$$T \le \left(\frac{\nu^{\frac{3}{4}}}{8\rho^{\frac{1}{2}}C\|u_0\|_{\dot{H}^{\frac{d}{2}-1}}}\right)^4. \tag{8.7}$$

So all terms in the r.h.s. of (8.6) have been made small enough s.t.

$$\|e^{\nu t\triangle}u_0\|_{L^4([0,T],\dot{H}^{\frac{d-1}{2}})} < \frac{\nu^{\frac{3}{4}}}{4C} \le \frac{1}{4\|B\|},$$

that is we obtained (8.4).

We have proved the 1st sentence in the statement of Theorem 7.2.

Now we turn to the proof that a solution $u \in L^4([0,T], \dot{H}^{\frac{d-1}{2}})$ satisfies (7.4).
By (8.1) we have $\mathcal{Q}_{NS}(u,u) \in L^2([0,T], \dot{H}^{\frac{d}{2}-2})$. Then it must be remarked that by its definition $B(u,u)$ is a solution in the sense of Definition 5.1 of the Heat Equation written above (7.2). Similarly, by Theorem 5.2 also $e^{\nu t \triangle} u_0$ is a solution of the homogeneous Heat Equation with initial value u_0. Hence, since u satisfies (7.2), then u is the solution of the Heat Equation (6.7), where the latter can be framed in terms of the theory in Sect. 5 for $s = \frac{d}{2} - 1$. Then by Theorem 5.2 we have $u \in C^0([0,T], \dot{H}^{\frac{d}{2}-1})$ and $\nabla u \in L^2([0,T], \dot{H}^{\frac{d}{2}})$. This yields (7.4).
We turn now to the proof of (7.7). We consider two solutions u and v, and set $w = u - v$. Then

$$\begin{cases} w_t - \nu \triangle w = \mathcal{Q}_{NS}(w, u+v) \\ \qquad w(0) = u_0 - v_0 \end{cases}$$

where we used the symmetry $\mathcal{Q}_{NS}(u,v) = \mathcal{Q}_{NS}(v,u)$ and

$$\mathcal{Q}_{NS}(u-v, u+v) = \mathcal{Q}_{NS}(u,u) - \mathcal{Q}_{NS}(v,v) + \underbrace{\mathcal{Q}_{NS}(u,v) - \mathcal{Q}_{NS}(v,u)}_{0}.$$

By the energy estimate (5.5) for $s = \frac{d}{2} - 1$ we have

$$\Delta_w := \|w(t)\|^2_{\dot{H}^{\frac{d}{2}-1}} + 2\nu \int_0^t \|\nabla w(t')\|^2_{\dot{H}^{\frac{d}{2}-1}} dt' = \|w_0\|^2_{\dot{H}^{\frac{d}{2}-1}} + 2 \int_0^t \langle \mathcal{Q}_{NS}(w, u+v), w \rangle_{\dot{H}^{\frac{d}{2}-1}}(t') dt'.$$

Claim 8.3. We have

$$\langle \mathcal{Q}_{NS}(a,b), c \rangle_{\dot{H}^{\frac{d}{2}-1}} \le C \|a\|_{\dot{H}^{\frac{d-1}{2}}} \|b\|_{\dot{H}^{\frac{d-1}{2}}} \|c\|_{\dot{H}^{\frac{d}{2}}}. \tag{8.8}$$

Proof. Indeed, trading derivatives we have

$$\langle \mathcal{Q}_{NS}(a,b), c \rangle_{\dot{H}^{\frac{d}{2}-1}} \le \|\mathcal{Q}_{NS}(a,b)\|_{\dot{H}^{\frac{d}{2}-2}} \|c\|_{\dot{H}^{\frac{d}{2}}}$$

and by (8.1) we have

$$\|\mathcal{Q}_{NS}(a,b)\|_{\dot{H}^{\frac{d}{2}-2}} \le C \|a\|_{\dot{H}^{\frac{d-1}{2}}} \|b\|_{\dot{H}^{\frac{d-1}{2}}}.$$

This proves Claim 8.3.

Now for $N(t) := \|u(t)\|_{\dot{H}^{\frac{d-1}{2}}} + \|v(t)\|_{\dot{H}^{\frac{d-1}{2}}}$ by Claim 8.3 we have

$$\Delta_w \le \|w_0\|^2_{\dot{H}^{\frac{d}{2}-1}} + 2 \int_0^t \|w(t')\|_{\dot{H}^{\frac{d-1}{2}}} N(t') \|\nabla w(t')\|_{\dot{H}^{\frac{d}{2}-1}} dt'.$$

By the interpolation estimate in Lemma 4.1 we have

$$\|w(t')\|_{\dot{H}^{\frac{d-1}{2}}} \le \|w(t')\|^{\frac{1}{2}}_{\dot{H}^{\frac{d}{2}-1}} \|\nabla w(t')\|^{\frac{1}{2}}_{\dot{H}^{\frac{d}{2}-1}}.$$

This implies

$$\Delta_w \leq \|w_0\|^2_{\dot{H}^{\frac{d}{2}-1}} + 2\int_0^t \|w(t')\|^{\frac{1}{2}}_{\dot{H}^{\frac{d}{2}-1}} N(t') \|\nabla w(t')\|^{\frac{3}{2}}_{\dot{H}^{\frac{d}{2}-1}} dt'.$$

Using the inequality $ab \leq \frac{1}{4}a^4 + \frac{3}{4}b^{\frac{4}{3}}$, which follows by

$$\log(ab) = \frac{1}{4}\log(a^4) + \frac{3}{4}\log(b^{\frac{4}{3}}) \leq \log\left(\frac{1}{4}a^4 + \frac{3}{4}b^{\frac{4}{3}}\right),$$

we get

$$\begin{aligned} \text{the integrand} &= \left(\|w(t')\|^{\frac{1}{2}}_{\dot{H}^{\frac{d}{2}-1}} N(t')\nu^{-\frac{3}{4}}\left(\frac{3}{4}\right)^{\frac{3}{4}}\right)\left(\frac{4}{3}\nu\|\nabla w(t')\|^2_{\dot{H}^{\frac{d}{2}-1}}\right)^{\frac{3}{4}} \\ &\leq \frac{C}{\nu^3}\|w(t')\|^2_{\dot{H}^{\frac{d}{2}-1}} N^4(t') + \nu\|\nabla w(t')\|^2_{\dot{H}^{\frac{d}{2}-1}}. \end{aligned}$$

Then

$$\Delta_w \leq \|w_0\|^2_{\dot{H}^{\frac{d}{2}-1}} + \frac{C}{\nu^3}\int_0^t \|w(t')\|^2_{\dot{H}^{\frac{d}{2}-1}} N^4(t')dt' + \nu\int_0^t \|\nabla w(t')\|^2_{\dot{H}^{\frac{d}{2}-1}} dt'.$$

In other words, by the definition of Δ_w

$$\begin{aligned} &\|w(t)\|^2_{\dot{H}^{\frac{d}{2}-1}} + \cancel{2}\nu\int_0^t \|\nabla w(t')\|^2_{\dot{H}^{\frac{d}{2}-1}} dt' \\ &\leq \|w_0\|^2_{\dot{H}^{\frac{d}{2}-1}} + \frac{C}{\nu^3}\int_0^t \|w(t')\|^2_{\dot{H}^{\frac{d}{2}-1}} N^4(t')dt' + \nu\cancel{\int_0^t \|\nabla w(t')\|^2_{\dot{H}^{\frac{d}{2}-1}} dt'} \end{aligned}$$

so that, if we set

$$X(t) := \|w(t)\|^2_{\dot{H}^{\frac{d}{2}-1}} + \nu\int_0^t \|\nabla w(t')\|^2_{\dot{H}^{\frac{d}{2}-1}} dt'$$

we have

$$X(t) \leq \|w_0\|^2_{\dot{H}^{\frac{d}{2}-1}} + \frac{C}{\nu^3}\int_0^t \|w(t')\|^2_{\dot{H}^{\frac{d}{2}-1}} N^4(t')dt'.$$

Hence

$$X(t) \leq \|w_0\|^2_{\dot{H}^{\frac{d}{2}-1}} + \frac{C}{\nu^3}\int_0^t X(t')N^4(t')dt'$$

and so by Gronwall's inequality

$$\|w(t)\|^2_{\dot{H}^{\frac{d}{2}-1}} + \nu\int_0^t \|\nabla w(t')\|^2_{\dot{H}^{\frac{d}{2}-1}} dt' \leq \|w_0\|^2_{\dot{H}^{\frac{d}{2}-1}} e^{\frac{C}{\nu^3}\int_0^t N^4(t')dt'}.$$

This proves the stability inequality (7.7)

We now consider the blow up criterion (7.5). Suppose that $u(t)$ is a solution in $[0,T)$ with

$$\int_0^T \|u(t)\|^4_{\dot H^{\frac{d-1}{2}}} dt < \infty.$$

Notice that then $u \in L^4([0,T], \dot H^{\frac{d-1}{2}})$ and

$$\|\mathcal{Q}_{NS}(u,u)\|_{L^2([0,T],\dot H^{\frac{d}{2}-2})} \le C\|u\|^2_{L^4([0,T],\dot H^{\frac{d-1}{2}}}. \tag{8.9}$$

We claim that we can extend $u(t)$ beyond T.

Claim 8.4. There exists a $\tau > 0$ s.t. u extends in a solution in $L^4([0,T+\tau), \dot H^{\frac{d-1}{2}}(\mathbb{R}^d, \mathbb{R}^d))$.

First of all we set

$$g(\xi) := \sup_{0\le t'\le T} |\widehat{u}(t',\xi)|.$$

Claim 8.5. We have $|\xi|^{\frac{d}{2}-1} g \in L^2(\mathbb{R}^d)$.

Proof of Claim 8.5. By (5.14) for $s = \frac{d}{2} - 1$ and by (8.1) we have

$$\begin{aligned}
\||\xi|^{\frac{d}{2}-1}g\|_{L^2} &= \left(\int_{\mathbb{R}^d} |\xi|^{d-2} \left(\sup_{0\le t'\le t} |\widehat{u}(t',\xi)|\right)^2 d\xi\right)^{\frac12} \\
&\le \|u_0\|_{\dot H^{\frac{d}{2}-1}} + \frac{1}{(2\nu)^{\frac12}} \|\mathcal{Q}_{NS}\|_{L^2([0,T],\dot H^{\frac{d}{2}-2})} \\
&\le \|u_0\|_{\dot H^{\frac{d}{2}-1}} + \frac{C}{(2\nu)^{\frac12}} \|u\|^2_{L^4([0,T],\dot H^{\frac{d-1}{2}})} < \infty.
\end{aligned}$$

This proves Claim 8.5.

Proof of Claim 8.4. Claim 8.5 implies

$$\int_{|\xi|\ge\rho} |\xi|^{d-2}|g(\xi)|^2 d\xi \overset{\rho\to\infty}{\to} 0.$$

Thus there exists $\rho > 0$ s.t for any preassigned $c > 0$

$$\int_{|\xi|\ge\rho} |\xi|^{d-2}|\widehat{u}(t,\xi)|^2 d\xi < (c\nu)^2 \text{ for all } t \in [0,T).$$

Now, recalling the splitting in high and low energies in the proof of the 1st sentence in the statement of Theorem 7.2, there exists a fixed $\tau > 0$ s.t. the lifespan of the solution with initial datum $u(t)$ is bounded below by τ independently of $t \in [0,T)$. Indeed there exists a $c_1 > 0$ independent from $t \in [0,T)$ s.t.

$$\left(\frac{\nu^{\frac34}}{8\rho^{\frac12} C\|u(t)\|_{\dot H^{\frac{d}{2}-1}}}\right)^4 > c_1 > 0.$$

This follows from the fact that

$$\|u(t)\|_{\dot H^{\frac{d}{2}-1}} \le \||\xi|^{\frac{d}{2}-1} g\|_{L^2} < \infty$$

So we can take $\tau = c_1$. Then $T_{u_0} \ge T + \tau$ and this yields Claim 8.4.

Let us now discuss the blow up criterion (7.6). Suppose that $T_{u_0} < \infty$ and that

$$C_{L2} := \int_0^{T_{u_0}} \|\nabla u(t)\|^2_{\dot H^{\frac{d}{2}-1}} dt < \infty. \tag{8.10}$$

Since we have (7.5) and

$$L^4([0,T],(\dot H^{\frac{d-1}{2}}(\mathbb{R}^d))^d) \subseteq L^\infty([0,T],(\dot H^{\frac{d}{2}-1}(\mathbb{R}^d))^d) \cap L^2([0,T],(\dot H^{\frac{d}{2}}(\mathbb{R}^d))^{d^2})$$

it follows that since we must have (7.5), then (8.10) implies that

$$\lim_{T\to T_{u_0}} \|u(t)\|_{L^\infty([0,T],\dot H^{\frac{d}{2}-1})} = \infty \tag{8.11}$$

For $0 \ge t \le T < T_{u_0}$ we have, by (8.8) and interpolation,

$$\begin{aligned}
&\|u(t)\|^2_{\dot H^{\frac{d}{2}-1}} + 2\nu \int_0^t \|\nabla u(t')\|^2_{\dot H^{\frac{d}{2}-1}} dt' = \|u(t_1)\|^2_{\dot H^{\sigma_s}} + 2\int_0^t \langle Q(u(t'),u(t')),u(t')\rangle_{\dot H^{\frac{d}{2}-1}} dt' \\
&\leqslant \|u(0)\|^2_{\dot H^{\frac{d}{2}-1}} + C'_d \int_0^t \|u(t')\|^2_{\dot H^{\frac{d-1}{2}}} \|\nabla u(t')\|_{\dot H^{\frac{d}{2}-1}} dt' \\
&\leqslant \|u(0)\|^2_{\dot H^{\frac{d}{2}-1}} + C_d \int_0^t \|u(t')\|_{\dot H^{\frac{d}{2}-1}} \|\nabla u(t')\|^2_{\dot H^{\frac{d}{2}-1}} dt'
\end{aligned} \tag{8.12}$$

and so

$$\|u\|^2_{L^\infty([0,T],\dot H^{\frac{d}{2}-1})} \leqslant \|u(0)\|^2_{\dot H^{\frac{d}{2}-1}} + C_d C_{L2} \|u\|_{L^\infty([0,T],\dot H^{\frac{d}{2}-1})}.$$

But this means that

$$\|u\|_{L^\infty([0,T],\dot H^{\frac{d}{2}-1})} \le \frac{1}{2} C_d C_{L2} + \frac{1}{2}\sqrt{C_d^2 C_{L2}^2 + 4\|u(0)\|^2_{\dot H^{\frac{d}{2}-1}}} < \infty,$$

in contradicting (8.11). This contradiction proves the blow up criterion (7.6).
The proof of Theorem 7.2 is completed. □

Corollary 8.6. *In the case $d = 2$, Theorem 7.2 implies Leray's Theorem 6.3 for $d = 2$*

Proof. By the Leray's Theorem 6.2 we know that given a divergence free $u_0 \in L^2(\mathbb{R}^2)$ there are weak solutions in the sense of Leray with $u \in L^\infty([0,\infty), L^2(\mathbb{R}^2,\mathbb{R}^2))$ and $\nabla u \in L^2([0,\infty), L^2(\mathbb{R}^2,\mathbb{R}^4))$. Interpolating, for each such a solution we have

$$\|\|u\|_{\dot H^{\frac{1}{2}}}\|_{L^4_t} \le \|\|u\|^{\frac{1}{2}}_{L^2} \|\nabla u\|^{\frac{1}{2}}_{L^2}\|_{L^4_t} \le \|u\|^{\frac{1}{2}}_{L^\infty_t L^2} \|\nabla u\|^{\frac{1}{2}}_{L^2_t L^2}$$

and so we obtain also $u \in L^4([0,\infty), \dot{H}^{\frac{1}{2}}(\mathbb{R}^2,\mathbb{R}^2))$.
By Lemma 8.2 we know that this implies

$$\mathcal{Q}_{NS}(u,u) \in L^2([0,\infty), \dot{H}^{-1}(\mathbb{R}^2,\mathbb{R}^2)).$$

Notice that the right hand side of (6.7) satisfies the hypothesis of the force term in the linear heat equation (5.1). As a weak solution of the Navier Stokes equation in the sense of Definition 6.1, u is then also a solution of the linear heat equation (5.1) in the sense of Definition 5.1. This means that it is also a solution of (7.2). Since by Theorem 7.2 such solution is a unique, we conclude that the solution of Leray's Theorem 6.2 in the case $d = 2$ is unique. Furthermore by Theorem 7.2 we know also that $u \in C^0([0,\infty), L^2(\mathbb{R}^2,\mathbb{R}^2))$.
We now turn to the energy identity. By Leray's Theorem 6.2 we know that

$$\|u(t)\|^2_{L^2(\mathbb{R}^2)} + 2\nu \int_0^t \|\nabla u(t')\|^2_{L^2(\mathbb{R}^2)} dt' \leq \|u_0\|^2_{L^2(\mathbb{R}^2)}.$$

We want now to prove that $\leq$ can be replaced by $=$ in this formula. As we have mentioned above, u solves in the sense of Definition 5.1 the problem

$$\partial_t u - \nu \triangle u = \mathcal{Q}_{NS}(u,u) \text{ with } \mathcal{Q}_{NS}(u,u) \in L^2(\mathbb{R}_+, \dot{H}^{-1}(\mathbb{R}^2,\mathbb{R}^2)),$$

Then, by Theorem 5.2 for $s = 0$ the identity (5.5) yields

$$\|u(t)\|^2_{L^2} + 2\nu \int_0^t \|\nabla u(t')\|^2_{L^2} dt' = \|u_0\|^2_{L^2} + 2\int_0^t \langle \mathcal{Q}_{NS}(u(t'),u(t')), u(t')\rangle_{L^2} dt'.$$

By Lemma 6.8 we have the cancelation

$$\langle \mathcal{Q}_{NS}(u,u), u\rangle = \langle \mathbb{P}(\text{div}(u\otimes u), u\rangle = \langle \text{div}(u\otimes u), u\rangle = 0.$$

This completes the proof, by giving the energy identity. □

8.1 Global solutions.

We start with the following lemma.

Lemma 8.7. *There exists $\varepsilon_1 > 0$ s.t. for $\|u_0\|_{\dot{H}^{\frac{d}{2}-1}} \leq \varepsilon_1$ the function $t \to \|u(t)\|_{\dot{H}^{\frac{d}{2}-1}}$ is decreasing.*

Proof. From Theorem 7.2 we know that for $\varepsilon_1 \in (0,\varepsilon_0]$ then we have $\|u(t)\|_{\dot{H}^{\frac{d}{2}-1}} \lesssim \|u_0\|_{\dot{H}^{\frac{d}{2}-1}} \leq \varepsilon_1$ for all t. Now, given any pair $0 \leq t_1 < t_2$ we have like in (8.12)

$$\|u(t_2)\|^2_{\dot{H}^{\frac{d}{2}-1}} + 2\nu \int_{t_1}^{t_2} \|\nabla u(t')\|^2_{\dot{H}^{\frac{d}{2}-1}} dt' \leqslant \|u(t_1)\|^2_{\dot{H}^{\frac{d}{2}-1}} + C\int_{t_1}^{t_2} \|u(t')\|_{\dot{H}^{\frac{d}{2}-1}} \|\nabla u(t')\|^2_{\dot{H}^{\frac{d}{2}-1}} dt'$$
$$\leqslant \|u(t_1)\|^2_{\dot{H}^{\frac{d}{2}-1}} + C\varepsilon_1 \int_{t_1}^{t_2} \|\nabla u(t')\|^2_{\dot{H}^{\frac{d}{2}-1}} dt',$$

where C is a fixed constant. Choosing ε_1 s.t. $C\varepsilon_1 < \nu$, it follows

$$\|u(t_2)\|^2_{\dot{H}^{\frac{d}{2}-1}} + \nu \int_{t_1}^{t_2} \|\nabla u(t')\|^2_{\dot{H}^{\frac{d}{2}-1}} dt' \leqslant \|u(t_1)\|^2_{\dot{H}^{\frac{d}{2}-1}}. \tag{8.13}$$

Hence $t \to \|u(t)\|_{\dot{H}^{\frac{d}{2}-1}}$ is decreasing. □

Proposition 8.8. *Let $d = 3$ and let $u_0 \in H^{\frac{1}{2}}(\mathbb{R}^3, \mathbb{R}^3)$ be s.t. $T_{u_0} = \infty$. Then*

$$\lim_{t\to+\infty} \|u(t)\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3,\mathbb{R}^3))} = 0. \tag{8.14}$$

Proof. Since $u_0 \in H^{\frac{1}{2}}(\mathbb{R}^3, \mathbb{R}^3))$ we have also $u_0 \in L^2(\mathbb{R}^3, \mathbb{R}^3))$, and u is also a weak solution in the sense of Leray. Hence it satisfies the energy inequality

$$\|u(t)\|^2_{L^2(\mathbb{R}^3)} + 2\nu \int_0^t \|\nabla u(t')\|^2_{L^2(\mathbb{R}^3)} dt' \leq \|u_0\|^2_{L^2(\mathbb{R}^3)},$$

which implies in particular

$$\|\nabla u\|_{L^2(\mathbb{R}_+, L^2(\mathbb{R}^3))} \leq \frac{1}{\sqrt{2\nu}} \|u_0\|_{L^2(\mathbb{R}^3)} \text{ and}$$
$$\|u\|_{L^\infty(\mathbb{R}_+, L^2(\mathbb{R}^3))} \leq \|u_0\|_{L^2(\mathbb{R}^3)}.$$

So by Hölder inequality and the interpolation of Lemma 4.1, we have

$$\|u\|_{L^4(\mathbb{R}_+, \dot{H}^{\frac{1}{2}}(\mathbb{R}^3))} \leq \frac{1}{\sqrt[4]{2\nu}} \|u_0\|_{L^2(\mathbb{R}^3)}.$$

This implies that for $1 \gg \epsilon > 0$ arbitrarily small, there exists $t_\epsilon > 0$ s.t. $\|u(t_\epsilon)\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)} \leq \epsilon$. So, in the half-line $[t_\epsilon, \infty)$ the function $u(t)$ is a small solution as of Theorem 7.2. But then $\|u(t)\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)} \leq \epsilon$ for all $t \geq t_\epsilon$ by Lemma 8.7, and, since $\epsilon > 0$ is arbitrary, we have the limit in (8.14). □

9 The case of initial data in $L^3(\mathbb{R}^3)$

It is possible to prove the following theorem.

Theorem 9.1. *For any divergence free $u_0 \in L^3(\mathbb{R}^3, \mathbb{R}^3)$ there is a $T > 0$ and a unique solution $u \in C^0([0, T), L^3(\mathbb{R}^3, \mathbb{R}^3))$ of*

$$u = e^{\nu t \triangle} u_0 + B(u, u). \tag{7.2}$$

Furthermore there exists a $\varepsilon_{3,\nu} > 0$ s.t. for $\|u_0\|_{L^3} < \varepsilon_{3,\nu}$ we have $T = \infty$. Furthermore, if $u_0 \in \dot{H}^{1/2}(\mathbb{R}^3, \mathbb{R}^3)$, the life span is the same of Theorem 7.2.

Exercise 9.2. Prove that the mapping $\dot{H}^{1/2}(\mathbb{R}^3, \mathbb{R}^3) \to L^3(\mathbb{R}^3, \mathbb{R}^3)$ is not surjective.

Exercise 9.3. Prove that the subspace of divergence free vector fields in $\dot{H}^{1/2}(\mathbb{R}^3, \mathbb{R}^3)$ is closed in $\dot{H}^{1/2}(\mathbb{R}^3, \mathbb{R}^3)$. Prove the same for with $\dot{H}^{1/2}(\mathbb{R}^3, \mathbb{R}^3)$ replaced by $L^3(\mathbb{R}^3, \mathbb{R}^3)$.

Exercise 9.4. Prove that the Sobolev embedding from the subspace of divergence free vector fields in $\dot{H}^{1/2}(\mathbb{R}^3, \mathbb{R}^3)$ to the subspace of divergence free vector fields in $L^3(\mathbb{R}^3, \mathbb{R}^3)$ is not surjective.

Exercise 9.5. Pick a divergence free u_0 belonging to $L^3(\mathbb{R}^3, \mathbb{R}^3)$ but not to $\dot{H}^{1/2}(\mathbb{R}^3, \mathbb{R}^3)$. Show that there exists a sequence of divergence free vector fields $\{u_0^{(n)}\}$ in $\dot{H}^{1/2}(\mathbb{R}^3, \mathbb{R}^3)$ with $u_0^{(n)} \to u_0$ in $L^3(\mathbb{R}^3, \mathbb{R}^3)$. Show also that $\|u_0^{(n)}\|_{\dot{H}^{1/2}} \to \infty$.

Exercise 9.6. Show that it is possible to define divergence free sequences $\{v_0^{(n)}\}$ in $\dot{H}^{1/2}(\mathbb{R}^3, \mathbb{R}^3)$ with $\|v_0^{(n)}\|_{\dot{H}^{1/2}} \to \infty$ and $\|v_0^{(n)}\|_{L^3} \to 0$.

Remark 9.7. For a sequence such as in Exercise 9.6, for $n \gg 1$ the corresponding solutions of the NS equation are globally defined in time by Theorem 9.14, while Theorem 7.2 is able to guarantee only on short intervals of time.

To prove Theorem 9.14 we will apply the abstract Lemma 7.1 in an appropriate Banach space X. The striking fact though, is that the space X will not be of the form $C^0([0,T], L^3(\mathbb{R}^3, \mathbb{R}^3))$. This because if X where this space, then the bilinear form B defined by (7.1) is known not to be continuous. It turns out that to get the right Banach space X, has required a certain degree of imagination and insight.

Definition 9.8. For $p \in [3, \infty]$ and $T \in (0, \infty)$ we set

$$K_p(T) = \{u \in C^0((0,T], L^p(\mathbb{R}^3, \mathbb{R}^3)) : \|u\|_{K_p(T)} := \sup_{t \in (0,T]} (\nu t)^{\frac{3}{2}\left(\frac{1}{3} - \frac{1}{p}\right)} \|u(t)\|_{L^p} < \infty\} \tag{9.1}$$

and for $p \in [1, 3)$

$$K_p(T) = \{u \in C^0([0,T], L^p(\mathbb{R}^3, \mathbb{R}^3)) : \|u\|_{K_p(T)} := \sup_{t \in (0,T]} (\nu t)^{\frac{3}{2}\left(\frac{1}{3} - \frac{1}{p}\right)} \|u(t)\|_{L^p} < \infty\}. \tag{9.2}$$

We denote by $K_p(\infty)$ the spaces defined as above, with $(0, T]$ replaced by $(0, \infty)$.

We recall that the solution of the heat equation $u_t - \nu \triangle u = 0$ is $e^{t\nu\triangle} f = K_t * f$ where $K_t(x) := (4\pi\nu t)^{-\frac{3}{2}} e^{-\frac{|x|^2}{4t\nu}}$. Notice that $K_t(x) = (\nu t)^{-\frac{3}{2}} K((\nu t)^{-\frac{1}{2}} x)$, where $K(x) := (4\pi)^{-\frac{d}{2}} e^{-\frac{|x|^2}{4}}$ and where $\widehat{K}(\xi) = e^{-|\xi|^2}$.

Notice that for $u_0 \in L^3(\mathbb{R}^3)$ and $p \geq 3$ we have from (1.15),

$$\|e^{t\nu\Delta} u_0\|_{L^p(\mathbb{R}^3)} \leqslant (4\pi\nu t)^{\frac{3}{2}\left(\frac{1}{p} - \frac{1}{3}\right)} \|u_0\|_{L^3(\mathbb{R}^3)} \text{ for all } p \geq 3, \tag{9.3}$$

it can be proved that $e^{t\nu\Delta} u_0 \in C(\mathbb{R}_+, L^p)$, and so $e^{t\nu\Delta} u_0 \in K_p(\infty)$.

Lemma 9.9. *Let $u_0 \in L^3(\mathbb{R}^3, \mathbb{R}^3)$ and $p > 3$. Then*

$$\lim_{T \to 0} \|e^{t\nu\Delta} u_0\|_{K_p(T)} = 0. \tag{9.4}$$

Proof. For any $\epsilon > 0$ there exists $\phi \in L^3(\mathbb{R}^3, \mathbb{R}^3) \cap L^p(\mathbb{R}^3, \mathbb{R}^3)$ s.t. $\|u - \phi\|_{L^3} < \epsilon$. Then by (9.3) we have

$$\|u - \phi\|_{K_p(T)} \leq (4\pi)^{\frac{3}{2}\left(\frac{1}{p} - \frac{1}{3}\right)} \epsilon.$$

Since $\|e^{t\nu\Delta}\phi\|_{L^p} \leq \|\phi\|_{L^p}$, it follows

$$\|e^{t\nu\triangle}\phi\|_{K_p(T)} = \sup_{t \in (0,T]} (\nu t)^{\frac{3}{2}\left(\frac{1}{3} - \frac{1}{p}\right)} \|e^{t\nu\Delta}\phi\|_{L^p} \leq (\nu T)^{\frac{3}{2}\left(\frac{1}{3} - \frac{1}{p}\right)} \|\phi\|_{L^p} \to 0 \text{ as } T \to 0.$$

□

Lemma 9.10. *Let p, q and r satisfy*

$$\begin{aligned} 0 < \frac{1}{p} + \frac{1}{q} \leqslant 1 \\ \frac{1}{r} \leqslant \frac{1}{p} + \frac{1}{q} < \frac{1}{3} + \frac{1}{r} \end{aligned} \tag{9.5}$$

Then the bilinear map B defined in (7.1) *maps $K_p(T) \times K_q(T) \to K_r(T)$ and there is a constant C independent from T s.t.*

$$\|B(u, v)\|_{K_r(T)} \leqslant C \|u\|_{K_p(T)} \|v\|_{K_q(T)}. \tag{9.6}$$

To prove Lemma 9.10 we consider for any $m = 1, 2, 3$ the problem

$$\begin{cases} (L_m f)_t - \nu \triangle L_m f = \mathbb{P} \partial_m f \\ L_m f(0, x) = 0 \end{cases} \tag{9.7}$$

($L_m f$ is by definition the solution of the above heat equation). Then by (10.8) and (6.9) for appropriate constants c_{jk} we have

$$\widehat{L_m f}(t, \xi) = \sum_{j,k=1}^{3} c_{jk} \int_0^t e^{-(t-t')\nu|\xi|^2} \xi_j \xi_k \xi_m |\xi|^{-2} \widehat{f}(t', \xi) dt'. \tag{9.8}$$

This means, for $\Gamma_{jkm}(t, x)$ the inverse Fourier transform of $e^{-t\nu|\xi|^2} \xi_j \xi_k \xi_m |\xi|^{-2}$,

$$L_m f(t) = \sum_{j,k=1}^{3} c_{jk} \int_0^t \Gamma_{jkm}(t - t') * \widehat{f}(t') dt'. \tag{9.9}$$

We claim the following.

Claim 9.11. We have for a fixed $C > 0$

$$|\Gamma_{jkm}(t,x)| \leqslant C(\sqrt{\nu t} + |x|)^{-4}. \tag{9.10}$$

Proof. It is elementary that $\Gamma_{jkm}(t,x) = (\nu t)^{-2}\Gamma_{jkm}((\nu t)^{-1/2}x)$ with $\widehat{\Gamma}_{jkm}(x) = e^{-|\xi|^2}\xi_j\xi_k\xi_m|\xi|^{-2}$. Then (9.10) is a consequence of

$$|\Gamma_{jkm}(x)| \leqslant C(1 + |x|)^{-4}. \tag{9.11}$$

Notice that $\Gamma_{jkm} \in C^\infty(\mathbb{R}^4) \cap L^\infty(\mathbb{R}^4)$, and this is straightforward by the rapid decay to 0 at infinity of $e^{-|\xi|^2}\xi_j\xi_k\xi_m|\xi|^{-2}$. Hence, to prove (9.11) it suffices to consider $|x| \gg 1$.For χ_0 a smooth cutoff of compact support equal to 1 near 0 and with $\chi_1 = 1 - \chi_0$, we set

$$\begin{aligned}\Gamma_{jkm}(x) &= (2\pi)^{-\frac{3}{2}} \int_{\mathbb{R}^3} e^{-\mathrm{i}\xi\cdot x}\chi_0\left(|x|\xi\right) e^{-|\xi|^2}\xi_j\xi_k\xi_m|\xi|^{-2}d\xi \\ &+ (2\pi)^{-\frac{3}{2}} \int_{\mathbb{R}^3} e^{-\mathrm{i}\xi\cdot x}\chi_1\left(|x|\xi\right) e^{-|\xi|^2}\xi_j\xi_k\xi_m|\xi|^{-2}d\xi\end{aligned}$$

The 1st term in the r.h.s. is

$$\lesssim \int_{|\xi|\le |x|^{-1}} |\xi|\ d\xi \sim |x|^{-4}.$$

For the other term we set $L := \mathrm{i}\frac{x}{|x|^2}\cdot\nabla_\xi$ noticing that $Le^{-\mathrm{i}\xi\cdot x} = e^{-\mathrm{i}\xi\cdot x}$. Then, the 2nd term is

$$(2\pi)^{-\frac{3}{2}} \int_{\mathbb{R}^3} e^{-\mathrm{i}\xi\cdot x}L^6\left(\chi_1\left(|x|\xi\right) e^{-|\xi|^2}\xi_j\xi_k\xi_m|\xi|^{-2}\right) d\xi.$$

The absolute value of the integrand is for fixed C

$$\left|L^6\left(\cdots\right)\right| \le C|x|^{-6}e^{-|\xi|^2}|\xi|^{-5}.$$

Here we used that in the support of $\nabla_\xi\left(\chi_1\left(|x|\xi\right)\right)$ we have $|x| \sim |\xi|^{-1}$. So the last integral is bounded

$$\lesssim |x|^{-6}\int_{1\ge|\xi|\ge|x|^{-1}} |\xi|^{-5}d\xi + |x|^{-6}\int_{|\xi|\ge 1} e^{-|\xi|^2}d\xi$$

where the 2nd term is $\sim |x|^{-6} \ll |x|^{-4}$ and the 1st term is $\sim |x|^{-6}|x|^2 = |x|^{-4}$. This completes the proof of Claim 9.11.

□

Completion of proof of Lemma 9.10. By (9.10) we have by Young's inequality for convolutions and Hölder's inequality for the tensor product of u and v the bound (here $\frac{1}{a} = 1 + \frac{1}{r} - \frac{1}{\beta}$ and $\frac{1}{\beta} = \frac{1}{p} + \frac{1}{q}$)

$$\begin{aligned}\|B(u,v)\|_{L^r} &\leqslant C_1 \sum_{j,m,k} \int_0^t \left\|\Gamma_{j,m,k}(t-t')\right\|_{L^a} \left\|u(t')\otimes v(t')\right\|_{L^\beta} dt' \\ &\leqslant C_1 \sum_{j,m,k} \int_0^t \left\|\Gamma_{j,m,k}(t-t')\right\|_{L^a} \left\|u(t')\right\|_{L^p} \left\|v(t')\right\|_{L^q} dt' \\ &\lesssim \int_0^t (t-t')^{-\frac{1}{2}-\frac{3}{2}\left(\frac{1}{p}+\frac{1}{q}-\frac{1}{r}\right)}(t')^{-\frac{3}{2}\left(\frac{2}{3}-\frac{1}{p}-\frac{1}{q}\right)} dt' \left\|u\right\|_{K_p(t)} \left\|v\right\|_{K_q(t)}\end{aligned}$$

where in the 3rd line we used

$$\left\|\Gamma_{j,m,k}(t-t')\right\|_{L^a(\mathbb{R}^3)} \lesssim \left\|(\sqrt{t-t'}+|x|)^{-4}\right\|_{L^a(\mathbb{R}^3)} = (t-t')^{-2}\| \left(1+\frac{|x|}{\sqrt{t-t'}}\right)^{-4} \|_{L^a(\mathbb{R}^3)}$$
$$= (t-t')^{-2}(t-t')^{\frac{3}{2a}} \left\|(1+|x|)^{-4}\right\|_{L^a(\mathbb{R}^3)} \sim (t-t')^{-2+\frac{3}{2}\left(1+\frac{1}{r}-\frac{1}{p}-\frac{1}{q}\right)}$$
$$= (t-t')^{-\frac{1}{2}-\frac{3}{2}\left(\frac{1}{p}+\frac{1}{q}-\frac{1}{r}\right)}.$$

We then conclude

$$\|B(u,v)\|_{L^r} \leqslant Ct^{-\frac{3}{2}\left(\frac{1}{3}-\frac{1}{r}\right)} \|u\|_{K_p(t)} \|v\|_{K_q(t)} \tag{9.12}$$

where we the fact that $\forall\ \alpha,\beta \in (-\infty,1)$ there is a $C(\alpha,\beta)\in\mathbb{R}_+$ s.t.

$$\int_0^t (t-t')^{-\alpha}(t')^{-\beta}dt' \leqslant C(\alpha,\beta)t^{1-\alpha-\beta} \text{ for all } t>0 \tag{9.13}$$

and

$$\frac{1}{2}+\frac{3}{2}\left(\frac{1}{p}+\frac{1}{q}-\frac{1}{r}\right)+\frac{3}{2}\left(\frac{2}{3}-\frac{1}{p}-\frac{1}{q}\right)=\frac{1}{2}+\frac{3}{2}\left(\frac{2}{3}-\frac{1}{r}\right)=\frac{1}{2}+1-\frac{3}{2r}$$
$$=2-\frac{1}{2}-\frac{3}{2r}=1+1-\frac{3}{2r}=1+\frac{3}{2}\left(\frac{1}{3}-\frac{1}{r}\right).$$

Notice that in the inequalities in (9.5) we need:

- $\frac{1}{\beta} := \frac{1}{p}+\frac{1}{q} \leqslant 1$ in order for $u\otimes v$ to belong to the Lebesgue space $L^\beta(\mathbb{R}^3)$;
- $0 < \frac{1}{p}+\frac{1}{q}$ is needed because otherwise in (9.12) we get $(t')^{-1}$ and the integral is undefined;
- $\frac{1}{r} \leqslant \frac{1}{p}+\frac{1}{q}$ is needed for $a \geq 1$;
- $\frac{1}{p}+\frac{1}{q} < \frac{1}{3}+\frac{1}{r}$ is needed to get $-\frac{1}{2}-\frac{3}{2}\left(\frac{1}{p}+\frac{1}{q}-\frac{1}{r}\right) > -1$ in the exponent of $(t-t')$ in (9.12).

□

Exercise 9.12. Prove (9.13). Hint, split the integral into sum of integrals in $[0,t/2]$ and $[t/2,t]$.

We have the following fact.

Proposition 9.13. *For any $p \in (3, \infty]$ there exists a constant $\varepsilon_{p\nu} > 0$ s.t. if*

$$\|e^{t\triangle}u_0\|_{K_p(T)} < \varepsilon_{p\nu} \tag{9.14}$$

then there exists and is unique u in the ball of center 0 and radius $2\varepsilon_{p\nu}$ in $K_p(T)$ which satisfies (7.2)*.*

Proof. Setting $r = q = p$, we see that for $p > 3$ we have $B : K_p(T) \times K_p(T) \to K_p(T)$ is bounded and with norm that admits a finite upper bound independent from T. The proof follows then from the abstract Lemma 7.1. □

Theorem 9.14. *For any $u_0 \in L^3(\mathbb{R}^3, \mathbb{R}^3)$ there is a $T > 0$ and solution $u \in C^0([0,T), L^3(\mathbb{R}^3, \mathbb{R}^3))$ of* (7.2) *which is unique. Furthermore there exists a $\varepsilon_3 > 0$ s.t. for $\|u_0\|_{L^3} < \varepsilon_{3\nu}$ we have $T = \infty$.*

Proof. We have $e^{t\triangle}u_0 \in K_p(T)$ for any $p > 3$, see (9.3). Furthermore, $\|e^{t\triangle}u_0\|_{K_p(T)} \overset{T\to 0^+}{\to} 0$ for $p > 3$ by Lemma 9.9. Then we can apply Proposition 9.13 concluding that there exists a solution u of (7.2) in $K_6(T)$ for $T > 0$ small enough. Applying Lemma 9.10 for $p = q = 6$ and $r = 3$ we get $B(u,u) \in C^0([0,T], L^3)$, and so $u \in C^0([0,T], L^3)$.
We assume now that there are two solutions u_1 and u_2. Setting $u_{21} = u_2 - u_1$ and $w_j = B(u_j, u_j)$ we have

$$\begin{cases} \partial_t u_{21} - \nu\triangle u_{21} = f_{21} \\ \quad u_{21}(0) = 0 \end{cases} \text{with}$$
$$f_{21} = 2Q(e^{\nu t\triangle}u_0, u_{21}) + Q(w_2, u_{21}) + Q(w_1, u_{21}).$$

By $L^{\frac{3}{2}}(\mathbb{R}^3) \hookrightarrow \dot{H}^{-\frac{1}{2}}(\mathbb{R}^3)$, which is the dual of Sobolev's Embedding $\dot{H}^{\frac{1}{2}}(\mathbb{R}^3) \hookrightarrow L^3(\mathbb{R}^3)$, we have

$$\|Q(u,v)\|_{\dot{H}^{-\frac{3}{2}}(\mathbb{R}^3)} \le \|u \otimes v\|_{\dot{H}^{-\frac{1}{2}}(\mathbb{R}^3)} \lesssim \|u \otimes v\|_{L^{\frac{3}{2}}(\mathbb{R}^3)} \le \|u\|_{L^3}\|v\|_{L^3}.$$

Then, by (5.5) and entering the definition of f_{21}

$$\begin{aligned} &\|u_{21}(t)\|^2_{\dot{H}^{-\frac{1}{2}}} + 2\nu \int_0^t \|\nabla u_{21}(t')\|^2_{\dot{H}^{-\frac{1}{2}}} dt' \le 4 \int_0^t \langle f(t'), u(t')\rangle_{\dot{H}^{-\frac{1}{2}}} dt' \\ &\le 2 \int_0^t \|Q(e^{\nu t'\triangle}u_0, u_{21})\|_{\dot{H}^{-\frac{3}{2}}} \|\nabla u_{21}(t')\|_{\dot{H}^{-\frac{1}{2}}} dt' \\ &+ 2 \int_0^t \|Q(w_2, u_{21}) + Q(w_1, u_{21})\|_{\dot{H}^{-\frac{3}{2}}} \|\nabla u_{21}(t')\|_{\dot{H}^{-\frac{1}{2}}} dt'. \end{aligned} \tag{9.15}$$

We bound the last line with for $j = 1, 2$

$$\begin{aligned} &2 \int_0^t \|Q(w_j, u_{21})\|_{\dot{H}^{-\frac{3}{2}}} \|\nabla u_{21}(t')\|_{\dot{H}^{-\frac{1}{2}}} dt' \lesssim \|w_j\|_{K_3(t)} \int_0^t \|u_{21}(t')\|_{L^3} \|\nabla u_{21}(t')\|_{\dot{H}^{-\frac{1}{2}}} dt' \\ &\lesssim \|w_j\|_{K_3(t)} \int_0^t \|\nabla u_{21}(t')\|^2_{\dot{H}^{-\frac{1}{2}}} dt', \end{aligned} \tag{9.16}$$

where in the last line we used Sobolev's Embedding $\dot{H}^{\frac{1}{2}}(\mathbb{R}^3) \hookrightarrow L^3(\mathbb{R}^3)$.
So, the last line of (9.15) is

$$\lesssim \left(\|w_1\|_{K_3(t)} + \|w_2\|_{K_3(t)}\right) \int_0^t \|\nabla u_{21}(t')\|^2_{\dot{H}^{-\frac{1}{2}}} dt'. \tag{9.17}$$

We split now

$$u_0 = u_0^{(1)} + u_0^{(2)} \text{ with } \|u_0^{(1)}\|_{L^3} < \epsilon \text{ and } u_0^{(2)} \in L^6 \cap L^3$$

and we bound similarly to (9.16)

$$\int_0^t \|Q(e^{\nu t'\triangle} u_0^{(1)}, u_{21})\|_{\dot{H}^{-\frac{3}{2}}} \|\nabla u_{21}(t')\|_{\dot{H}^{-\frac{1}{2}}} dt' \lesssim \|u_0^{(1)}\|_{L^3} \int_0^t \|\nabla u_{21}(t')\|^2_{\dot{H}^{-\frac{1}{2}}} dt'.$$

Finally, we bound

$$\begin{aligned}
&\int_0^t \|Q(e^{\nu t'\triangle} u_0^{(2)}, u_{21})\|_{\dot{H}^{-\frac{3}{2}}} \|\nabla u_{21}(t')\|_{\dot{H}^{-\frac{1}{2}}} dt' \\
&\leq \int_0^t \|e^{\nu t'\triangle} u_0^{(2)} \otimes u_{21}\|_{\dot{H}^{-\frac{1}{2}}} \|\nabla u_{21}(t')\|_{\dot{H}^{-\frac{1}{2}}} dt' \lesssim \int_0^t \|e^{\nu t'\triangle} u_0^{(2)} \otimes u_{21}\|_{L^{\frac{3}{2}}} \|\nabla u_{21}(t')\|_{\dot{H}^{-\frac{1}{2}}} dt' \\
&\leq \int_0^t \|e^{\nu t'\triangle} u_0^{(2)}\|_{L^6} \; \|u_{21}\|_{L^2} \|\nabla u_{21}\|_{H^{\frac{1}{2}}} dt' \leq \|u_0^{(2)}\|_{L^6} \int_0^t \|u_{21}\|^{\frac{1}{2}}_{H^{-\frac{1}{2}}} \|\nabla u_{21}\|^{\frac{3}{2}}_{H^{\frac{1}{2}}} dt'.
\end{aligned}$$

So we get

$$\begin{aligned}
&\|u_{21}(t)\|^2_{\dot{H}^{-\frac{1}{2}}} + 2\nu \int_0^t \|\nabla u_{21}(t')\|^2_{\dot{H}^{-\frac{1}{2}}} dt' \lesssim \left(\|w_1\|_{K_3(t)} + \|w_2\|_{K_3(t)} + \|u_0^{(1)}\|_{L^3}\right) \int_0^t \|\nabla u_{21}(t')\|^2_{\dot{H}^{-\frac{1}{2}}} dt \\
&+ \frac{3}{4\mathbf{C}^{\frac{4}{3}}} \int_0^t \|\nabla u_{21}(t')\|^2_{\dot{H}^{-\frac{1}{2}}} dt' + \frac{\mathbf{C}^4}{4} \|u_0^{(2)}\|^4_{L^6} \int_0^t \|u_{21}\|^2_{H^{-\frac{1}{2}}} dt'.
\end{aligned}$$

Taking $\mathbf{C}$ large, and t small, so that $\|w_1\|_{K_3(t)} + \|w_2\|_{K_3(t)} + \|u_0^{(1)}\|_{L^3} < 3\epsilon$ with ϵ sufficiently small, we obtain

$$\|u_{21}(t)\|^2_{\dot{H}^{-\frac{1}{2}}} + \nu \int_0^t \|\nabla u_{21}(t')\|^2_{\dot{H}^{-\frac{1}{2}}} dt' \lesssim \frac{\mathbf{C}^4}{4} \|u_0^{(2)}\|^4_{L^6} \int_0^t \|u_{21}\|^2_{H^{-\frac{1}{2}}} dt'.$$

Gronwall's Inequality implies that $u_{21}(t') = 0$ for $t' \in [0, t]$ with $t > 0$ sufficiently small.
The above argument shows that the set

$$\{t \in [0, T) : u_{21} \equiv 0 \text{ in } [0, t]\}$$

is open (and, obviously, non empty) in $[0, T)$. On the other hand, since $u_{21} \in C^0([0, T), L^3(\mathbb{R}^3, \mathbb{R}^3))$ it is also closed in $[0, T)$. Hence it coincides with $[0, T)$.

□

Remark 9.15. Let $u_0 \in \dot H^{\frac12}(\mathbb{R}^3,\mathbb{R}^3)$. Then it can be proved that if $T_3 > 0$ is the lifespan of the corresponding solution $u \in C^0([0,T_3), L^3(\mathbb{R}^3,\mathbb{R}^3))$ provided by Theorem 9.14 and if $T_{u_0} > 0$ is the lifespan of the solution provided by Theorem 7.2, we have $T_3 = T_{u_0}$. We will prove the simpler result in Proposition 9.16.

Proposition 9.16. *Let $u_0 \in \dot H^{\frac12}(\mathbb{R}^3,\mathbb{R}^3)$. There there exists $\epsilon_{3\nu} > 0$ s.t. for $\|u_0\|_{L^3(\mathbb{R}^3)} < \epsilon_{3\nu}$ and if $T_{u_0} > 0$ is the lifespan of the solution provided by Theorem 7.2, we have $T_{u_0} = \infty$.*

Proof. Taking $\epsilon_{3\nu} > 0$ sufficiently small we can assume by Theorem 9.14 that $u \in C^0([0,\infty), L^3)$. In fact, if it is sufficiently small we can prove $\|u\|_{L^\infty([0,\infty),L^3)} < C_\nu \|u_0\|_{L^3}$ for a fixed $C_\nu > 0$. Suppose that $T_{u_0} < \infty$. Then by Theorem 7.2 we have the blow up

$$\lim_{T \nearrow T_{u_0}} \int_0^T \|\nabla u(t)\|^2_{\dot H^{\frac12}} dt = \infty. \tag{9.18}$$

By Theorem 7.2 and by (5.5), for $0 < t \le T < T_{u_0}$ we have

$$\|u(t)\|^2_{\dot H^{\frac12}} + 2\nu \int_0^t \|\nabla u(t')\|^2_{\dot H^{\frac12}} dt' = \|u_0\|^2_{\dot H^{\frac12}} + 2\int_0^t \langle u(t')\cdot \nabla u(t'), u(t')\rangle_{\dot H^{\frac12}} dt'. \tag{9.19}$$

By Sobolev's Embedding $\dot H^{\frac12}(\mathbb{R}^3,\mathbb{R}^3) \hookrightarrow L^3(\mathbb{R}^3,\mathbb{R}^3)$ we obtain

$$|\langle u\cdot\nabla u, u\rangle_{\dot H^{\frac12}}| = |\langle u\cdot\nabla u, \nabla u\rangle_{L^2}| \le \|u\|_{L^3}\|\nabla u\|^2_{L^3} \le C\|u\|_{L^3}\|\nabla u\|^2_{\dot H^{\frac12}}.$$

Then

$$\begin{aligned}&\|u(t)\|^2_{\dot H^{\frac12}} + 2\nu \int_0^t \|\nabla u(t')\|^2_{\dot H^{\frac12}} dt' \le \|u_0\|^2_{\dot H^{\frac12}} + C\|u\|_{L^\infty(\mathbb{R}_+,L^3)} \int_0^t \|\nabla u(t')\|^2_{\dot H^{\frac12}} dt' \\ &\le \|u_0\|^2_{\dot H^{\frac12}} + C_\nu C\|u_0\|_{L^3} \int_0^t \|\nabla u(t')\|^2_{\dot H^{\frac12}} dt'.\end{aligned}$$

So, for $C_\nu C\|u_0\|_{L^3} < \nu$, we get

$$\|u(t)\|^2_{\dot H^{\frac12}} + \nu \int_0^t \|\nabla u(t')\|^2_{\dot H^{\frac12}} dt' \le \|u_0\|^2_{\dot H^{\frac12}},$$

which contradicts (9.18).

□

10 Schrödinger equations

For $u_0 \in \mathcal{S}'(\mathbb{R}^d,\mathbb{C})$ the linear homogeneous Schrödinger equation is

$$\mathrm{i}u_t + \triangle u = 0\,, \quad u(0,x) = u_0(x).$$

By applying $\mathcal{F}$ we transform the above problem into

$$\widehat{u}_t + \mathrm{i}|\xi|^2\widehat{u} = 0\,, \quad \widehat{u}(0,\xi) = \widehat{u}_0(\xi).$$

This yields $\widehat{u}(t,\xi) = e^{-\mathrm{i}t|\xi|^2}\widehat{u}_0(\xi)$. We have $e^{-\mathrm{i}t|\xi|^2} = \widehat{G}(t,\xi)$ with $G(t,x) = (2t\mathrm{i})^{-\frac{d}{2}}e^{\frac{\mathrm{i}|x|^2}{4t}}$. This follows from the following generalization of (1.2) for $\mathrm{Re}\, z > 0$

$$e^{-z\frac{|\xi|^2}{2}} = (2\pi z)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{-\mathrm{i}\xi\cdot x} e^{-\frac{|x|^2}{2z}} dx.$$

This formula follows from the fact that both sides are holomorphic in $\mathrm{Re}\, z > 0$ and coincide for $z \in \mathbb{R}_+$. Then taking the limit $z \to 2\mathrm{i}$ for $\mathrm{Re}\, z > 0$ and using the continuity of $\mathcal{F}$ in $\mathcal{S}'(\mathbb{R}^d,\mathbb{C})$ we get

$$e^{-\mathrm{i}|\xi|^2} = (4\pi\mathrm{i})^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{-\mathrm{i}\xi\cdot x} e^{\frac{\mathrm{i}|x|^2}{4}} dx.$$

Then $u(t,x) = (2\pi)^{-\frac{d}{2}} G(t,\cdot) * u_0(x)$. In particular, for $u_0 \in L^p(\mathbb{R}^d,\mathbb{C})$ for $p \in [1,2]$ and by Reisz's interpolation defines for any $t > 0$ an operator which we denote by

$$e^{\mathrm{i}\triangle t}u_0(x) = (4\pi\mathrm{i}t)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{\frac{\mathrm{i}|x-y|^2}{4t}} u_0(y)dy \tag{10.1}$$

which is s.t. $e^{\mathrm{i}\triangle t} : L^p(\mathbb{R}^d,\mathbb{C}) \to L^{p'}(\mathbb{R}^d,\mathbb{C})$ for $p \in [1,2]$ and $p' = \frac{p}{p-1}$ with $\|e^{\mathrm{i}\triangle t}u_0\|_{L^{p'}} \le (4\pi t)^{-d(\frac{1}{2}-\frac{1}{p'})}\|u_0\|_{L^p}$ by Riesz interpolation.

Remark 10.1. Notice that for no $p \neq 2$ and $t > 0$ we have that $e^{\mathrm{i}\triangle t}$ defines a bounded operator $L^p(\mathbb{R}^d,\mathbb{C}) \to L^p(\mathbb{R}^d,\mathbb{C})$, see [9].

Remark 10.2. Notice that $e^{\triangle t} : L^p(\mathbb{R}^d) \to L^q(\mathbb{R}^d)$ is a bounded operator for all $1 \le p \le q \le \infty$.

In the sequel, given $v, w \in L^2(\mathbb{R}^d,\mathbb{C})$ we will use the notation

$$\langle v, w\rangle = \mathrm{Re} \int_{\mathbb{R}^d} v(x)\overline{w}(x)dx.$$

We consider

$$\mathrm{i}u_t + \triangle u = f\,, \quad u(0) = u_0 \in H^1(\mathbb{R}^d). \tag{10.2}$$

Definition 10.3. Let $s \in \mathbb{R}$. For $f \in L^1([0,T], H^{s-2}(\mathbb{R}^d))$ we say that

$$u \in L^\infty([0,T], H^s(\mathbb{R}^d)), \tag{10.3}$$

is a weak solution of (10.2) if

$$u \text{ is weakly continuous from } [0,T] \text{ into } H^s(\mathbb{R}^d) \tag{10.4}$$

(that is, if for any $\psi \in H^{-s}(\mathbb{R}^d)$ the function $t \to \langle u(t),\psi\rangle$, which is a well defined function in $L^\infty([0,T],\mathbb{R})$, is in fact in $C^0([0,T],\mathbb{R})$)

and if for any $\Psi \in C_c^\infty([0,T] \times \mathbb{R}^d)$ we have

$$\langle u(t), \Psi(t)\rangle_{L^2} = \int_0^t \left(\langle -\mathrm{i}u(t'), \triangle\Psi(t')\rangle_{L^2} + \langle u(t'), \partial_t\Psi(t')\rangle_{L^2} + \langle -\mathrm{i}f(t'), \Psi(t')\rangle_{L^2} \right) dt' + \langle u_0, \Psi(0)\rangle_{L^2}. \tag{10.5}$$

If also

$$u \in C^0([0,T], H^s(\mathbb{R}^d)), \tag{10.6}$$

we say that u is a strong solution of (10.2).

Theorem 10.4. *In Problem* (10.2) *assume $f \in L^1([0,T], H^s(\mathbb{R}^d))$. Then there is exactly one solution in the sense of the above definition. For any t we have:*

$$\|u(t)\|_{H^s(\mathbb{R}^d)} \le \|u_0\|_{H^s(\mathbb{R}^d)} + \|f\|_{L^1(0,t),H^s(\mathbb{R}^d))}. \tag{10.7}$$

Furthermore, the solution is strong, that is $u \in C^0([0,T], H^s(\mathbb{R}^d))$, and the following formula holds

$$\widehat{u}(t,\xi) = e^{\mathrm{i}t|\xi|^2}\widehat{u}_0(\xi) - \mathrm{i}\int_0^t e^{\mathrm{i}(t-t')|\xi|^2}\widehat{f}(t',\xi)dt'. \tag{10.8}$$

Proof. The proof is similar to that of Theorem 5.2 and is skipped.

□

Notice that (10.8) can be written as

$$u(t) = e^{-\mathrm{i}t\triangle}u_0 - \mathrm{i}\int_0^t e^{-\mathrm{i}(t-t')\triangle}f(t')dt'. \tag{10.9}$$

We say that a pair (q,r) is *admissible* when

$$\begin{aligned} &\frac{2}{q} = d\left(\frac{1}{2} - \frac{1}{r}\right) \\ &2 \le r \le \frac{2d}{d-2} \ (2 \le r \le \infty \text{ if } d=1,\ 2 \le r < \infty \text{ if } d=2). \end{aligned} \tag{10.10}$$

The pair $(\infty, 2)$ is always admissible. The *endpoint* $(2, \frac{2d}{d-2})$ is admissible for $d \ge 3$. We have the following important result.

Theorem 10.5 (Strichartz's estimates)**.** *The following facts hold.*

(1) For every $u_0 \in L^2(\mathbb{R}^d)$ we have $e^{\mathrm{i}\triangle t}u_0 \in L^q(\mathbb{R}, L^r(\mathbb{R}^d)) \cap C^0(\mathbb{R}, L^2(\mathbb{R}^d))$ for every admissible (q,r). Furthermore, there exists a C s.t.

$$\|e^{\mathrm{i}\triangle t}u_0\|_{L^q(\mathbb{R},L^r(\mathbb{R}^d))} \le C\|u_0\|_{L^2}. \tag{10.11}$$

(2) *Let I be an interval and let $t_0 \in \overline{I}$. If (γ,ρ) is an admissible pair and $f \in L^{\gamma'}(I, L^{\rho'}(\mathbb{R}^d))$ then for any admissible pair (q,r) the function*

$$\mathcal{T}f(t) = \int_{t_0}^{t} e^{\mathrm{i}\triangle(t-s)} f(s)ds \tag{10.12}$$

belongs to $L^q(I, L^r(\mathbb{R}^d)) \cap C^0(\overline{I}, L^2(\mathbb{R}^d))$ and there exists a constant C independent of I and f s.t.

$$\|\mathcal{T}f\|_{L^q(I,L^r(\mathbb{R}^d))} \le C\|f\|_{L^{\gamma'}(I,L^{\rho'}(\mathbb{R}^d))}. \tag{10.13}$$

Proof. The proof is skipped. See the very readable account in [8]. □

11 The semilinear Schrödinger equation

We will consider pure power semilinear Schrödinger equations

$$\begin{cases} \mathrm{i}u_t = -\triangle u + \lambda |u|^{p-1}u \text{ for } (t,x) \in [0,\infty) \times \mathbb{R}^d \\ u(0,x) = u_0(x) \end{cases} \tag{11.1}$$

for $\lambda \in \mathbb{R}\backslash\{0\}$ and $p > 1$. Here $p < d^*$ with $d^* = \infty$ for $d = 1, 2$ and $d^* = \frac{d+2}{d-2}$ for $d \ge 3$. We collect here a number of facts needed later.

Lemma 11.1. *We have the following facts.*

(1) *For $1 < p < d^*$ we have the Gagliardo–Nirenberg inequality:*

$$\|u\|_{L^{p+1}(\mathbb{R}^d)} \le C_p \|\nabla u\|^{\alpha}_{L^2(\mathbb{R}^d)} \|u\|^{1-\alpha}_{L^2(\mathbb{R}^d)} \text{ for } \frac{1}{p+1} = \frac{1}{2} - \frac{\alpha}{d}. \tag{11.2}$$

(2) *The map $u \to |u|^{p-1}u$ is a locally Lipschitz from $H^1(\mathbb{R}^d)$ to $H^{-1}(\mathbb{R}^d)$.*

Proof. For (1) see Theorem 4.2.
We turn (2). By (11.2) we know that $u \to |u|^{p-1}u$ maps $H^1(\mathbb{R}^d) \to L^{p+1}(\mathbb{R}^d) \to L^{\frac{p+1}{p}}(\mathbb{R}^d)$. Furthermore this map is locally Lipschitz:

$$\begin{aligned} &\||u|^{p-1}u - |v|^{p-1}v\|_{L^{\frac{p+1}{p}}} \le C\|(|u|^{p-1} + |v|^{p-1})(u-v)\|_{L^{\frac{p+1}{p}}} \\ &\le C'(\|u\|^p_{L^{p+1}} + \|v\|^p_{L^{p+1}})\|u-v\|_{L^{p+1}} \end{aligned}$$

where we have used, for $w = v - u$,

$$|u|^{p-1}u - |v|^{p-1}v = \int_0^1 \frac{d}{dt}\left(|u+tw|^{p-1}(u+tw)\right)dt =$$
$$\int_0^1 |u+tw|^{p-1}dtw + \int_0^1 (u+tw)\frac{d}{dt}\left((u_1+tw_1)^2 + (u_2+tw_2)^2\right)^{\frac{p-1}{2}} dt = \int_0^1 |u+tw|^{p-1}dtw +$$
$$\sum_{j=1}^{2}\int_0^1 (u+tw)\frac{p-1}{2}\left((u_1+tw_1)^2 + (u_2+tw_2)^2\right)^{\frac{p-3}{2}} 2(u_j+tw_j)dtw_j$$

which from $|u+tw| \leq |u|+|v|$ for $t \in [0,1]$ and

$$\left|(u+tw)\frac{p-1}{2}\left((u_1+tw_1)^2+(u_2+tw_2)^2\right)^{\frac{p-3}{2}} 2(u_j+tw_j)w_j\right| \leq (p-1)|u+tw|^{p-1}|w|$$

yields

$$\left||u|^{p-1}u-|v|^{p-1}v\right| \leq p(|u|+|v|)^{p-1}|u-v| \leq p2^{p-1}(|u|^{p-1}+|v|^{p-1})|u-v|,$$

where in the last step we used, for $|u| \geq |v|$,

$$(|u|+|v|)^{p-1} \leq 2^{p-1}|u|^{p-1} \leq 2^{p-1}(|u|^{p-1}+|v|^{p-1}).$$

Next, we show that we have an embedding $L^{\frac{p+1}{p}}(\mathbb{R}^d) \hookrightarrow H^{-1}(\mathbb{R}^d)$. Indeed, this is equivalent to $H^1(\mathbb{R}^d) \hookrightarrow L^{p+1}(\mathbb{R}^d)$ with in turn is a consequence of (11.2). □

We introduce now the following quantities:

$$\begin{aligned} E(u) &= \frac{1}{2}\int_{\mathbb{R}^d} |\nabla u|^2 dx + \frac{\lambda}{p+1}\int_{\mathbb{R}^d} |u|^{p+1} dx \\ Q(u) &= \int_{\mathbb{R}^d} |u|^2 dx. \end{aligned} \tag{11.3}$$

Here $E(u)$ is the energy, and is well defined in $H^1(\mathbb{R}^d)$ and $Q(u)$ is the mass or charge.

Remark 11.2. Notice, passingly, that $Q \in C^\infty(H^1(\mathbb{R}^d), \mathbb{R})$ while $E \in C^1(H^1(\mathbb{R}^d), \mathbb{R})$.

Definition 11.3 (Weak and strong solutions)**.** On some interval $(-S,T)$ a function

$$u \in L^\infty_{loc}((-S,T), H^1(\mathbb{R}^d)) \cap W^{1,\infty}((-S,T), H^{-1}(\mathbb{R}^d)) \tag{11.4}$$

is a *weak solution* of (11.1) if $u(0)=u_0$ in $H^{-1}(\mathbb{R}^d)$ and if for any $\Psi \in C_c^\infty((-S,T)\times\mathbb{R}^d)$ we have

$$\int_{-S}^{T} \left(\langle \mathrm{i}\triangle u(t') - \mathrm{i}|u(t')|^{p-1}u(t'), \Psi(t')\rangle + \langle u(t'), \partial_t \Psi(t')\rangle\right) dt' = 0. \tag{11.5}$$

u is a *strong solution* if furthermore

$$u \in C^0((-S,T), H^1(\mathbb{R}^d)) \cap C^1((-S,T), H^{-1}(\mathbb{R}^d)). \tag{11.6}$$

Definition 11.4 (Well posedness in $H^1(\mathbb{R}^d)$)**.** We say that the problem (11.1) is locally well posed if the following facts hold:

1. For any $u_0 \in H^1(\mathbb{R}^d)$ there exists and is unique a maximal strong solution u.

2. All weak solutions are strong solutions.

3. Consider the lifespan $(-S, T)$ of a maximal solution u and suppose that $T < \infty$. Then

$$\lim_{t \nearrow T} \|u(t)\|_{H^1} = +\infty \tag{11.7}$$

   with an analogous formula if $S < +\infty$.

4. Suppose $u_{0n} \to u_0$ in $H^1(\mathbb{R}^d)$ and consider their corresponding maximal solutions

$$u_n \in C^0((-S_n, T_n), H^1(\mathbb{R}^d)) \cap C^1((-S_n, T_n), H^{-1}(\mathbb{R}^d)).$$

   Consider $[-a, b] \subset (-S, T)$. Then

$$\lim u_n = u \text{ in } C^0([-a, b], H^1(\mathbb{R}^d)) \cap C^1([-a, b]), H^{-1}(\mathbb{R}^d)). \tag{11.8}$$

The problem is *globally well posed* if it is locally well posed and the lifespan of all solutions is $\mathbb{R}$.

Theorem 11.5 (Local well posedness in $H^1(\mathbb{R}^d)$)**.** *For $0 < p < d^*$ the problem* (11.1) *is locally well posed. Furthermore, the functions $E(u(t))$, $Q(u(t))$ are constant.*

12 Proof of Theorem 11.5

We first prove the existence of some weak solutions.

12.1 Existence of some weak solutions

Proposition 12.1 (Local existence of weak solutions)**.** *For any $u_0 \in H^1(\mathbb{R}^d)$ there exists a weak solution of* (11.1) *in $(-T_1(u_0), T_2(u_0))$, with $T_j(u_0) > 0$. Furthermore, we have $Q(u(t)) = Q(u_0)$ and $E(u(t)) \le E(u_0)$.*

Proof. The proof consist of two main steps:

1. We consider a sequence of approximating ODE's, we prove existence of a corresponding of approximate solutions $\{u_n\}$ and some bonds for the u_n. We show that, up to a subsequence, the sequence $\{u_n\}$ has a limit u.

2. The most delicate part of the proof consists in proving that the limit u is a weak solution of (11.1)

Step 1: truncations of the NLS. For $\varphi \in C_c^\infty(\mathbb{R}, [0, 1])$ a function with $\varphi = 1$ near 0 and with support contained in the ball $B_{\mathbb{R}^d}(0, r_0)$, consider [2] the operators $\mathbf{Q}_n =$

[2] Notice that using everywhere the projections $\mathbf{P}_n = \chi_{[0,n]}(\sqrt{-\triangle})$ would be a bad choice for this proof. Difficulties would arise from the fact proved by C.Feffermann [6] that $\mathbf{P}_n$ for $d \ge 2$ is bounded from $L^p(\mathbb{R}^d)$ into itself only if $p = 2$. On the other hand it is elementary that the $\mathbf{Q}_n$ are of the form $\rho_{\frac{1}{n}} *$ for a $\rho \in \mathcal{S}(\mathbb{R}^d)$ and so are uniformly bounded from $L^p(\mathbb{R}^d)$ into itself for all p and form a sequence converging strongly to the identity operator.

$\varphi(\sqrt{-\triangle}/n)$. The truncations $\mathbf{Q}_n(|u|^{p-1}u)$ are locally Lipschitz functions from $H^1(\mathbb{R}^d)$ into itself as they are compositions $H^1(\mathbb{R}^d) \overset{|u|^{p-1}u}{\to} H^{-1}(\mathbb{R}^d) \overset{\mathbf{Q}_n}{\to} H^1(\mathbb{R}^d))$ of a locally Lipschitz function, Lemma 11.1, and of bounded linear maps.
We consider the following truncations of the NLS

$$\begin{cases} \mathrm{i}u_{nt} = -\mathbf{P}_{nr_0}\triangle u_n + \lambda \mathbf{Q}_n(|\mathbf{Q}_n u_n|^{p-1}\mathbf{Q}_n u_n) \text{ for } (t,x) \in \mathbb{R}\times\mathbb{R}^d \\ u_n(0) = \mathbf{Q}_n u_0 \end{cases} \tag{12.1}$$

By the theory of ODE's, there exists a maximal solution $u_n(t) \in C^1(-T_1(n), T_2(n)), H^1(\mathbb{R}^d))$ of (12.1) . Furthermore, if $T_2(n) < \infty$ then we must have blow up

$$\lim_{t \nearrow T_2(n)} \|u_n(t)\|_{H^1} = +\infty \text{ if } T_2(n) < \infty \tag{12.2}$$

with a similar blow up phenomenon if $T_1(n) < \infty$.
To get bounds on this sequence of functions we search for invariants of motion. First of all we apply $\langle \cdot, \mathrm{i}u_n\rangle$ to (12.1) and get

$$\frac{1}{2}\frac{d}{dt}\|u_n(t)\|_{L^2} = -\langle \mathbf{P}_{nr_0}\triangle u_n, \mathrm{i}u_n\rangle + \lambda\langle \mathbf{Q}_n(|\mathbf{Q}_n u_n|^{p-1}\mathbf{Q}_n u_n), \mathrm{i}u_n\rangle. \tag{12.3}$$

We show in a moment that both terms in the r.h.s. are equal to 0. This implies

$$\|u_n(t)\|_{L^2} = \|\mathbf{Q}_n u_0\|_{L^2} \le \|u_0\|_{L^2}. \tag{12.4}$$

To prove that the r.h.s. of (12.3) is 0, observe that the the 1st term is 0 because the bounded operator $\mathrm{i}\mathbf{P}_{nr_0}\triangle$ of $L^2(\mathbb{R}^d)$ into itself is antisymmetric: $(\mathrm{i}\mathbf{P}_{nr_0}\triangle)^* = -\mathrm{i}\mathbf{P}_{nr_0}\triangle$. For the 2nd term we use

$$\langle \mathbf{Q}_n(|\mathbf{Q}_n u_n|^{p-1}\mathbf{Q}_n u_n), \mathrm{i}u_n\rangle = \langle |\mathbf{Q}_n u_n|^{p-1}\mathbf{Q}_n u_n, \mathrm{i}\mathbf{Q}_n u_n\rangle = \lambda \operatorname{Re} \mathrm{i} \int_{\mathbb{R}^d} |\mathbf{Q}_n u_n|^{p+1} dx = 0.$$

This yields (12.4). Now we consider the energy functional associated to (12.1). Applying $\langle \cdot, u_{nt}\rangle$ to (12.1)

$$\begin{aligned} 0 &= -\langle \mathbf{P}_{nr_0}\triangle u_n, u_{nt}\rangle + \lambda\langle \mathbf{Q}_n(|\mathbf{Q}_n u_n|^{p-1}\mathbf{Q}_n u_n), u_{nt}\rangle = -\langle \triangle u_n, u_{nt}\rangle + \lambda\langle |\mathbf{Q}_n u_n|^{p-1}\mathbf{Q}_n u_n, \mathbf{Q}_n u_{nt}\rangle \\ &= \frac{d}{dt}\underbrace{(\frac{1}{2}\|\nabla u_n\|_{L^2}^2 + \frac{\lambda}{p+1}\int_{\mathbb{R}^d} |\mathbf{Q}_n u_n|^{p+1} dx)}_{E_n(u_n)} = \frac{d}{dt}E_n(u_n). \end{aligned}$$

where we used the fact, easy to check, that $u_n = \mathbf{P}_{nr_0}u_n$. Hence

$$E_n(u_n(t)) = E_n(\mathbf{Q}_n u_0). \tag{12.5}$$

(12.4) implies $T_1(n) = T_2(n) = \infty$. Indeed (12.2) does not hold because

$$\|u_n(t)\|_{H^1} = \|\mathbf{P}_{nr_0}u_n(t)\|_{H^1} \le nr_0\|u_n(t)\|_{L^2} = nr_0\|\mathbf{Q}_n u_0\|_{L^2} \le nr_0\|u_0\|_{L^2}. \tag{12.6}$$

Let us now fix M such that $\|u_0\|_{H^1} < M$ and let us set

$$\theta_n := \sup\{\tau > 0 : \|u_n(t)\|_{H^1} < 2M \text{ for } |t| < \tau.\} \tag{12.7}$$

Then by Lemma 11.1

$$\|u_{nt}\|_{L^\infty((-\theta_n,\theta_n),H^{-1})} < C(M).$$

Our main focus is now to prove that there exists a fixed $T(M) > 0$ s.t. $\theta_n \geq T(M)$ for all n.

First of all we prove that $u_n \in C^{0,\frac{1}{2}}((-\theta_n,\theta_n), L^2)$ with a fixed Hölder constant $C(M)$. By an interpolation similar to Lemma 4.1

$$\begin{aligned} &\|u_n(t) - u_n(s)\|_{L^2} \lesssim \|u_n(t) - u_n(s)\|_{H^1}^{\frac{1}{2}} \|u_n(t) - u_n(s)\|_{H^{-1}}^{\frac{1}{2}} \\ &\leq \sqrt{2}\|u_n\|_{L^\infty((-\theta_n,\theta_n),H^1)}^{\frac{1}{2}} \|u_{nt}\|_{L^\infty((-\theta_n,\theta_n),H^{-1})}^{\frac{1}{2}} \sqrt{|t-s|} \\ &\leq C(M)\sqrt{|t-s|} \text{ for } t,s \in (-\theta_n,\theta_n) \end{aligned} \tag{12.8}$$

Now we want to prove

$$\|u_n(t)\|_{H^1}^2 \leq \|u_0\|_{H^1}^2 + C(M)t^b \text{ for some fixed } b > 0 \text{ and for } t \in (-\theta_n,\theta_n). \tag{12.9}$$

From $E_n(u_n(t)) = E_n(\mathbf{Q}_n u_0)$ and $Q(u_n(t)) = Q(\mathbf{Q}_n u_0)$ we get

$$\|u_n(t)\|_{H^1}^2 + \frac{2\lambda}{p+1}\int_{\mathbb{R}^d} |\mathbf{Q}_n u_n|^{p+1} dx = \|\mathbf{Q}_n u_0\|_{H^1}^2 + \frac{2\lambda}{p+1}\int_{\mathbb{R}^n} |\mathbf{Q}_n^2 u_0|^{p+1} dx.$$

Hence using Hölder and Gagliardo–Nirenberg

$$\begin{aligned} &\|u_n(t)\|_{H^1}^2 \leq \|u_0\|_{H^1}^2 + \frac{2|\lambda|}{p+1}\int_{\mathbb{R}^d} \left| \ |\mathbf{Q}_n u_n(t)|^{p+1} - |\mathbf{Q}_n^2 u_0|^{p+1} \right| dx \\ &\leq \|u_0\|_{H^1}^2 + C\int_{\mathbb{R}^d} (|\mathbf{Q}_n u_n(t)|^p + |\mathbf{Q}_n^2 u_0|^p)|\mathbf{Q}_n u_n(t) - \mathbf{Q}_n^2 u_0| dx \\ &\leq \|u_0\|_{H^1}^2 + C\||\mathbf{Q}_n u_n(t)|^p + |\mathbf{Q}_n^2 u_0|^p\|_{L^{\frac{p+1}{p}}} \|\mathbf{Q}_n u_n(t) - \mathbf{Q}_n^2 u_0\|_{L^{p+1}} \\ &\leq \|u_0\|_{H^1}^2 + C_1 \left(\|\mathbf{Q}_n u_n(t)\|_{L^{p+1}}^p + \|\mathbf{Q}_n^2 u_0\|_{L^{p+1}}^p\right) \|u_n(t) - \mathbf{Q}_n u_0\|_{\dot{H}^1}^{\alpha} \|u_n(t) - \mathbf{Q}_n u_0\|_{L^2}^{1-\alpha} \end{aligned}$$

Then by (12.8) with $s = 0$, the Sobolev Embedding Theorem and (12.7) we get (12.9).
Now for $T(M)$ defined s.t. $C(M)T(M)^b = 2M^2$ (for the $C(M)$ in (12.9)) from (12.9) we get

$$\|u_n(t)\|_{L^\infty([-T(M),T(M)],H^1)} \leq \sqrt{3}M. \tag{12.10}$$

Since $\sqrt{3}M < 2M$ this obviously means that $T(M) < \theta_n$ since, if we had $\theta_n \leq T(M)$ then, by the fact that $u_n \in C^1(\mathbb{R}, H^1)$, the definition of θ_n in (12.7) would be contradicted.
Hence we have

$$\|u_n\|_{L^\infty([-T(M),T(M)],H^1)} < 2M \tag{12.11}$$

and, by (12.1), also

$$\|u_{nt}\|_{L^\infty([-T(M),T(M)],H^{-1})} < C(M). \tag{12.12}$$

Our next claim is about the existence of a limit u of the sequence.

Claim 12.2. There exists u with

$$\|u\|_{L^\infty([-T(M),T(M)],H^1(\mathbb{R}^d))} < 2M \tag{12.13}$$

s.t. up to a subsequence $u_n(t) \rightharpoonup u(t)$ in $H^1(\mathbb{R}^d))$ for all $t \in [-T(M), T(M)]$.

Proof. $\{u_n\}_n$ is a bounded sequence in $C^1([-T(M),T(M)], H^{-1}(\mathbb{R}^d))$ by (12.6), by $T(M) \in (0, \theta_n)$ and by (12.10). Up to a subsequence, $u_n(t) \rightharpoonup u(t)$ in $H^{-1}(\mathbb{R}^d)$ for all $t \in \mathbb{Q} \cap [-T(M), T(M)]$. It is not restrictive to assume that the subsequence coincides with the sequence. It is then easy to conclude, using equicontinuity, that in fact $\{u_n(t)\}_n$ is weakly convergent in $H^{-1}(\mathbb{R}^d)$ for all $t \in [-T(M), T(M)]$. By the lower semicontinuity of the norm for the weak topology, the equicontinuity of the sequence $\{u_n\}_n$ implies that $u \in C^0([-T(M),T(M)], H^{-1}(\mathbb{R}^d))$. Recall that $u_n \in C^0([-T(M),T(M)], H^1)$ and that we have (12.11), in fact the better estimate (12.10). So by weak compactness it is easy to conclude that for all $t \in [-T(M), T(M)]$ we have $u(t) \in H^1$ with $u_n(t) \rightharpoonup u(t)$ in $H^1(\mathbb{R}^d)$. Indeed, if this was false there would be a $t \in [-T(M), T(M)]$ and a subsequence $u_{n_k}(t) \rightharpoonup v$ in $H^1(\mathbb{R}^d)$ with $v \neq u(t)$. But this is impossible because we know that we must have $v = u(t)$ in $H^{-1}(\mathbb{R}^d)$
By the lower semicontinuity of the norm for the weak topology, $u \in L^\infty([-T(M), T(M)], H^1(\mathbb{R}^d))$ with (12.13) satisfied. □

Step 2: u is a weak solution of (11.1)**.** First of all, we need to show that u solves an equation. So let us see what happens to the the sequence of nonlinear terms $\mathbf{Q}_n(|\mathbf{Q}_n u_n|^{p-1}\mathbf{Q}_n u_n)$ of (12.1).

Claim 12.3. $\{|\mathbf{Q}_n u_n(t)|^{p-1}\mathbf{Q}_n u_n(t)\}_n$ forms a bounded sequence in $C^{0,a}([-T(M),T(M)], L^{\frac{p+1}{p}})$ for some $a > 0$.

Proof. We have, using Gagliardo–Nirenberg (11.2) and (12.8),

$$\begin{aligned}
&\||\mathbf{Q}_n u_n(t)|^{p-1}\mathbf{Q}_n u_n(t) - |\mathbf{Q}_n u_n(s)|^{p-1}\mathbf{Q}_n u_n(s)\|_{L^{\frac{p+1}{p}}} \\
&\lesssim \|(|\mathbf{Q}_n u_n(t)|^{p-1} + |\mathbf{Q}_n u_n(s)|^{p-1})(\mathbf{Q}_n u_n(t) - \mathbf{Q}_n u_n(s))\|_{L^{\frac{p+1}{p}}} \\
&\lesssim (\|u_n(t)\|^{p-1}_{L^{p+1}} + \|u_n(t)\|^{p-1}_{L^{p+1}})\|u_n(t) - u_n(s))\|_{L^{p+1}} \\
&\leq C_1(M)\|u_n(t) - u_n(s)\|^{\alpha}_{H^1(\mathbb{R}^d)}\|u_n(t) - u_n(s)\|^{1-\alpha}_{L^2(\mathbb{R}^d)} \leq C(M)|t-s|^a,
\end{aligned}$$

for some $a > 0$. □

By the Claim 12.3 and proceeding like for Claim 12.2 up to a subsequence we have

$$|\mathbf{Q}_n u_n(t)|^{p-1}\mathbf{Q}_n u_n(t) \rightharpoonup f(t) \text{ in } L^{\frac{p+1}{p}}(\mathbb{R}^d, \mathbb{C}) \text{ for all } t \in [-T(M), T(M)] \tag{12.14}$$

and

$$f \in C^{0,a}([-T(M),T(M)], L^{\frac{p+1}{p}}) \subset C^{0,a}([-T(M),T(M)], H^{-1})$$

On the other hand, for every $w \in C_c^\infty(\mathbb{R}^d, \mathbb{C})$ and any $\varphi \in C_c^\infty((-T(M), T(M)), \mathbb{R})$, we have

$$\int_{-T(M)}^{T(M)} \left[-\langle \mathrm{i}u_n, w\rangle \dot\varphi(t) + \langle \mathbf{P}_{nr_0}\triangle u_n + \mathbf{Q}_n(|\mathbf{Q}_n u_n|^{p-1}\mathbf{Q}_n u_n), w\rangle \varphi(t)\right] dt = 0.$$

Taking the limit, by the strong limits $\mathbf{P}_{nr_0} w \to w$ and $\mathbf{Q}_n w \to w$ we have

$$\int_{-T(M)}^{T(M)} \left[-\langle \mathrm{i}u, w\rangle \dot\varphi(t) + \langle \triangle u + f, w\rangle \varphi(t)\right] dt = 0.$$

This implies the distributional equality in $(-T(M), T(M))$

$$\mathrm{i}u_t = -\triangle u + f. \tag{12.15}$$

This implies $u \in W^{1,\infty}([-T(M), T(M)], H^{-1})$ which, in turn, implies that $u \in C^0([-T(M), T(M)], H^{-1})$ and since $u_n(0) = \mathbf{Q}_n u_0 \overset{n\to\infty}{\to} u_0$ and $u_n(0) \overset{n\to\infty}{\rightharpoonup} u(0)$ we have $u(0) = u_0$. So we proved that $u \in W^{1,\infty}([-T(M), T(M)], H^{-1})$ and that

$$\begin{cases} \mathrm{i}u_t = -\triangle u + f \\ \quad u(0) = u_0. \end{cases}$$

Now we need to show that $f = \lambda |u|^{p-1}u$. However before proving this, we prove another claim in Proposition 12.1.

Claim 12.4. We have

$$\|u(t)\|_{L^2} = \|u_0\|_{L^2}. \tag{12.16}$$

Proof. We start by showing that for all $t \in [-T(M), T(M)]$ we have $\mathrm{Im}[f(t)\overline{u}(t)] = 0$ a.e. in $\mathbb{R}^d$. It suffices to show , for any ball $B \subset \mathbb{R}^d$, that

$$\langle f(t), \mathrm{i}u(t)\rangle_{L^2(B)} = 0.$$

We have (ignoring λ)

$$\begin{aligned} &\langle f, \mathrm{i}u\rangle_{L^2(B)} = \langle f - |\mathbf{Q}_n u_n|^{p-1}\mathbf{Q}_n u_n, \mathrm{i}u\rangle_{L^2(B)} + \langle |\mathbf{Q}_n u_n|^{p-1}\mathbf{Q}_n u_n, \mathrm{i}(1-\mathbf{Q}_n)u\rangle_{L^2(B)} \\ &+ \langle |\mathbf{Q}_n u_n|^{p-1}\mathbf{Q}_n u_n, \mathrm{i}\mathbf{Q}_n(u-u_n)\rangle_{L^2(B)} + \langle |\mathbf{Q}_n u_n|^{p-1}\mathbf{Q}_n u_n, \mathrm{i}\mathbf{Q}_n u_n\rangle_{L^2(B)} \to a + b + c + 0. \end{aligned}$$

We have $a = 0$ since

$$\langle f - |\mathbf{Q}_n u_n|^{p-1}\mathbf{Q}_n u_n), \mathrm{i}u\rangle_{L^2(B)} \to 0 \text{ by } f - |\mathbf{Q}_n u_n|^{p-1}\mathbf{Q}_n u_n \rightharpoonup 0 \text{ in } L^{\frac{p+1}{p}}(\mathbb{R}^d) \subset H^{-1}(\mathbb{R}^d).$$

We have $b = 0$ since

$$\begin{aligned} &|\langle |\mathbf{Q}_n u_n|^{p-1}\mathbf{Q}_n u_n, \mathrm{i}(\mathbf{Q}_n - 1)u\rangle_{L^2(B)}| \le \||\mathbf{Q}_n u_n|^{p-1}\mathbf{Q}_n u_n\|_{L^{\frac{p+1}{p}}(\mathbb{R}^d)} \|(\mathbf{Q}_n - 1)u\|_{L^{p+1}(\mathbb{R}^d)} \\ &\le \|\mathbf{Q}_n u_n\|_{L^{p+1}}^p \|(\mathbf{Q}_n - 1)u\|_{L^{p+1}(\mathbb{R}^d)} \\ &\lesssim \|u_n\|_{L^{p+1}}^p \|(\mathbf{Q}_n - 1)u\|_{L^{p+1}(\mathbb{R}^d)} \le C(M)\|(\mathbf{Q}_n - 1)u\|_{H^1(\mathbb{R}^d)} \to 0 \end{aligned}$$

by the strong limit $\mathbf{Q}_n u \to u$ in H^1.
We now show that $c = 0$. First of all we have

$$|\langle |\mathbf{Q}_n u_n|^{p-1}\mathbf{Q}_n u_n, \mathrm{i}\mathbf{Q}_n(u-u_n)\rangle| \le C(M)\|\mathbf{Q}_n(u-u_n)\|_{L^{p+1}(B)}.$$

Next, we have $\mathbf{Q}_n(u-u_n) \rightharpoonup 0$ in $H^1(\mathbb{R}^d)$. Since the map $H^1(\mathbb{R}^d) \overset{v\to v|_B}{\to} L^{p+1}(B)$ is compact and

$$\|\mathbf{Q}_n(u-u_n)\|_{L^{p+1}(B)} \le \|\mathbf{Q}_n(u-u_n)\|_{L^{p+1}(\mathbb{R}^d)} \le C\|u-u_n\|_{H^1(\mathbb{R}^d)} \le 4C\ C(M)$$

it follows that $\mathbf{Q}_n(u-u_n) \to 0$ in $L^{p+1}(B)$ and hence $c = 0$.
Hence we conclude that for all $t \in [-T(M), T(M)]$ we have $\mathrm{Im}[f(t)\overline{u}(t)] = 0$ a.e. in $\mathbb{R}^d$.
Now we prove the conservation of mass (12.16).
Apply $\langle \cdot, \mathrm{i}\mathbf{P}_R u(t)\rangle$ to the equation of u. We get

$$\langle u_t, \mathbf{P}_R u\rangle = \langle f(t), \mathrm{i}\mathbf{P}_R u\rangle.$$

Notice that $\mathbf{P}_R u_t = (\mathbf{P}_R u)_t$ with $\mathbf{P}_R u \in W^{1,\infty}((-T(M), T(M)), H^1(\mathbb{R}^d))$. Then $\|\mathbf{P}_R u\|^2_{L^2} \in W^{1,\infty}((-T(M), T(M)), \mathbb{R})$ with

$$\frac{d}{dt}\|\mathbf{P}_R u\|^2_{L^2} = 2\langle u_t, \mathbf{P}_R u\rangle = 2\langle f(t), \mathrm{i}\mathbf{P}_R u\rangle.$$

Hence

$$|\|\mathbf{P}_R u(t)\|^2_{L^2} - \|\mathbf{P}_R u\|^2_{L^2}| \le \int_0^t 2|\langle(\mathbf{P}_R - 1)f(t'), \mathrm{i}u(t')\rangle| dt' \overset{R\to+\infty}{\to} 0.$$

This completes the proof of the conservation of mass (12.16). □

Now we turn to the proof of $f = \lambda|u|^{p-1}u$.
First of all by (12.13) and by inequalities like (12.8) it follows that

$$u \in C^{0,\frac{1}{2}}([-T(M), T(M)], L^2(\mathbb{R}^d, \mathbb{C})). \tag{12.17}$$

Recall from Clam 12.2 that $u_n(t) \rightharpoonup u(t)$ in $H^1(\mathbb{R}^d))$ for all $t \in [-T(M), T(M)]$. Now we claim the following.

Claim 12.5. We have

$$u_n \to u \text{ in } L^\infty([-T(M), T(M)], L^2(\mathbb{R}^d)). \tag{12.18}$$

Proof. We proceed by contradiction. If (12.18) is false there is a sequence t_n s.t. $\|u(t_n) - u_n(t_n)\|^2_{L^2} \ge \varepsilon > 0$. Then, up to a subsequence, we get $t_n \to \bar{t}$. We claim that we have the following limit, which contradicts $\|u(t_n) - u_n(t_n)\|^2_{L^2} \ge \varepsilon > 0$:

$$\begin{aligned} &\|u(t_n) - u_n(t_n)\|^2_{L^2} = \|u(t_n)\|^2_{L^2} + \|u_n(t_n)\|^2_{L^2} - 2\langle u_n(\bar{t}), u(\bar{t})\rangle \\ &- 2\langle u_n(t_n) - u_n(\bar{t}), u(\bar{t})\rangle - 2\langle u_n(t_n), u(t_n) - u(\bar{t})\rangle \to 0 \text{ as } n \to \infty. \end{aligned} \tag{12.19}$$

To see why the limit holds, notice that the first line on the r.h.s $\to 0$. Indeed $\|u(t)\|_{L^2} = \|u_0\|_{L^2}$, $\|u_n(t)\|_{L^2} = \|\mathbf{Q}_n u_0\|_{L^2} \overset{n\to\infty}{\to} \|u_0\|_{L^2}$ and by Claim 12.2 we have $-2\langle u_n(\bar{t}), u(\bar{t})\rangle \overset{n\to\infty}{\to} -2\|u(\bar{t})\|^2_{L^2}$.
Next we show that also the 2nd line of (12.19) $\to 0$. First of all $u(t_n) - u(\bar{t}) \to 0$ in $L^2(\mathbb{R}^d)$ by $u \in C^0([-T(M), T(M)], L^2(\mathbb{R}^d))$, see (12.17). We also have

$$|\langle u_n(t_n) - u_n(\bar{t}), u(\bar{t})\rangle| \le |t_n - \bar{t}|^{\frac{1}{2}} \|u_n\|_{L^\infty(-T(M),T(M)),H^{-1})} \|u(\bar{t})\|_{H^1} \le C(M)|t_n - \bar{t}|^{\frac{1}{2}} \to 0.$$

by (12.12). Therefore (12.19) and Claim 12.5 are proved. □
By Gagliardo–Nirenberg, (11.2) and by (12.11), (12.13) and (12.18) we conclude

$$u_n \to u \text{ in } C^0((-T(M), T(M)), L^{p+1}(\mathbb{R}^d)). \tag{12.20}$$

Since $|\mathbf{Q}_n u_n(t)|^{p-1}\mathbf{Q}_n u_n(t) \rightharpoonup f(t)$ in $L^{\frac{p+1}{p}}(\mathbb{R}^d)$ by (12.14) and $|u_n(t)|^{p-1}u_n(t) \to |u(t)|^{p-1}u(t)$ in $L^{\frac{p+1}{p}}(\mathbb{R}^d)$ by (12.20), the following claim implies $f = \lambda|u|^{p-1}u$.

Claim 12.6. We have

$$\begin{aligned} &\mathbf{Q}_n(|\mathbf{Q}_n u_n|^{p-1}\mathbf{Q}_n u_n) - |u_n|^{p-1}u_n = \mathbf{Q}_n[|\mathbf{Q}_n u_n|^{p-1}\mathbf{Q}_n u_n - |\mathbf{Q}_n u|^{p-1}\mathbf{Q}_n u] \\ &+ \mathbf{Q}_n[|\mathbf{Q}_n u|^{p-1}\mathbf{Q}_n u - |u|^{p-1}u] + (\mathbf{Q}_n - 1)(|u|^{p-1}u) \to 0 \text{ in } L^{\frac{p+1}{p}}(\mathbb{R}^d). \end{aligned} \tag{12.21}$$

Proof. The claim follows from the following remarks.

- We have

$$\begin{aligned} &\|\mathbf{Q}_n(|\mathbf{Q}_n u_n|^{p-1}\mathbf{Q}_n u_n - |\mathbf{Q}_n u|^{p-1}\mathbf{Q}_n u)\|_{L^{\frac{p+1}{p}}} \\ &\le \||\mathbf{Q}_n u_n|^{p-1}\mathbf{Q}_n u_n - |\mathbf{Q}_n u|^{p-1}\mathbf{Q}_n u\|_{L^{\frac{p+1}{p}}} \\ &\lesssim \|(|\mathbf{Q}_n u_n|^{p-1} + |\mathbf{Q}_n u|^{p-1})(\mathbf{Q}_n u_n - \mathbf{Q}_n u)\|_{L^{\frac{p+1}{p}}} \\ &\lesssim (\|u_n\|^{p-1}_{L^{p+1}} + \|u\|^{p-1}_{L^{p+1}})\|u_n - u\|_{L^{p+1}} \le C(M)\|u_n - u\|_{L^{p+1}(\mathbb{R}^d)} \to 0. \end{aligned}$$

- We have by $\mathbf{Q}_n - 1 \to 0$ in the strong sense of bounded operators in $L^q(\mathbb{R}^d)$ for any $q \in (1, \infty)$

$$\begin{aligned} &\|\mathbf{Q}_n(|\mathbf{Q}_n u|^{p-1}\mathbf{Q}_n u - |u|^{p-1}u)\|_{L^{\frac{p+1}{p}}} \\ &\le \||\mathbf{Q}_n u|^{p-1}\mathbf{Q}_n u - |u|^{p-1}u\|_{L^{\frac{p+1}{p}}} \\ &\lesssim \|(|\mathbf{Q}_n u|^{p-1} + |u|^{p-1})(\mathbf{Q}_n u - u)\|_{L^{\frac{p+1}{p}}} \\ &\lesssim (\|\mathbf{Q}_n u\|^{p-1}_{L^{p+1}} + \|u\|^{p-1}_{L^{p+1}})\|(\mathbf{Q}_n - 1)u\|_{L^{p+1}} \le 2\|u\|^{p-1}_{L^{p+1}}\|(\mathbf{Q}_n - 1)u\|_{L^{p+1}(\mathbb{R}^d)} \to 0. \end{aligned}$$

- $|u|^{p-1}u \in L^{\frac{p+1}{p}}(\mathbb{R}^d)$ implies $(\mathbf{Q}_n - 1)(|u|^{p-1}u) \to 0$.

Hence Claim 12.6 is proved. □

To complete the proof of Proposition 12.1 it remains to be shown that $E(u(t)) \le E(u_0)$. Recall that for

$$E_n(v) := \frac{1}{2}\|\nabla v\|_{L^2}^2 + \frac{\lambda}{p+1}\int_{\mathbb{R}^n} |\mathbf{Q}_n v|^{p+1} dx$$

we have shown $E_n(u_n) = E_n(\mathbf{Q}_n u_0)$. Now we have

- $\|\nabla u\|_{L^2}^2 \le \liminf \|\nabla u_n\|_{L^2}^2$.
- $\lim \mathbf{Q}_n u_0 = u_0$ in $H^1(\mathbb{R}^d)$.
- $u - \mathbf{Q}_n u_n = (1-\mathbf{Q}_n)u + \mathbf{Q}_n(u - u_n) \to 0$ in $L^{p+1}(\mathbb{R}^d)$.

This implies $E(u) \le \liminf E_n(u_n) = \lim E_n(\mathbf{Q}_n u_0) = E(u_0)$. Hence the proof of Proposition 12.1 is completed.

□

12.2 Well posedness assuming uniqueness

First of all, assuming uniqueness we get well posedness. The 1st step is the proof of the conservation of energy, which is a consequence of the *time reversibility* of the NLS.

Proposition 12.7. *Suppose that we know that the solutions in Proposition 12.1 are unique. Then $E(u(t_0)) = E(u_0)$ for any t_0 and (11.1) is well posed.*

Proof. Also $v(t,x) := \overline{u}(t_0 - t, x)$ is a solution of the equation. Since $E(u_0) = E(v(t_0)) \le E(v(0)) = E(u(t_0))$ we get the opposite inequality to $E(u(t_0)) \le E(u_0)$ and so we conclude with the energy equality.

Now we show that the energy conservation implies that u is a strong solution. We know already that in its lifespan $(-S,T)$ we have

$$u \in C^{0,\frac{1}{2}}((-S,T), L^2(\mathbb{R}^d)) \cap C^0((-S,T), L^{p+1}(\mathbb{R}^d)).$$

In particular we know that $t \to \int_{\mathbb{R}^d} |u(t)|^{p+1} dx$ is in $C^0((-S,T),\mathbb{R})$. By $E(u(t)) = E(u_0)$ and $Q(u(t)) = Q(u_0)$ we conclude that also $t \to \|u(t)\|_{H^1}$ is in $C^0((-S,T),\mathbb{R})$. It is easy to see that

$$u \in C^0((-S,T), H^1(\mathbb{R}^d)) \text{ with } H^1(\mathbb{R}^d) \text{ endowed with the weak topology.} \tag{12.22}$$

This is equivalent at proving $\langle u, \psi\rangle \in C^0((-S,T),\mathbb{R})$ for any $\psi \in H^1(\mathbb{R}^d)$. This can be seen by the weak limit $u_n(t) \rightharpoonup u(t)$ in $H^1(\mathbb{R}^d))$ like in Claim 12.2 for $t \in [a,b]$ with $[a,b] \subset (-S,T)$ an appropriate compact interval of any preassigned $t_0 \in (-S,T)$. Then $\langle u_n(t), \psi\rangle \to \langle u(t), \psi\rangle$ for $t \in [a,b]$ with $[a,b] \subset (-S,T)$. Furthermore, bounds like (12.12) imply that the sequence $\{\langle u_n(t), \psi\rangle\}$ is equicontinuous. This implies that $\langle u, \psi\rangle \in C^0([a,b],\mathbb{R})$ for any $\psi \in H^1(\mathbb{R}^d)$ and proves (12.22).

(12.22) and $\|u(t)\|_{H^1} \in C^0((-S,T),\mathbb{R})$ imply $u \in C^0((-S,T),H^1(\mathbb{R}^d))$ with $H^1(\mathbb{R}^d)$ endowed with the strong topology. Since u solves (11.1) it follows that (11.6) is true, that is that we have also $u \in C^1((-S,T),H^{-1}(\mathbb{R}^d))$.
Now we prove that if $T < \infty$ then

$$\lim_{t \nearrow T} \|\nabla u(t)\|_{L^2(\mathbb{R}^d)} = +\infty$$

Indeed, if there is a sequence $t_j \nearrow T$ s.t. $\|u(t_j)\|_{H^1(\mathbb{R}^d)} \le M < \infty$, then one can extend $u(t)$ beyond $t_j + T(M) > T$ and get a contradiction.
Now we show continuity in terms of the initial data. Let $u_0^{(n)} \to u_0$ in H^1. Fix $[-t_1,t_2] \subset (-S,T)$ and set

$$M := 2\sup\{\|u(t)\|_{H^1(\mathbb{R}^d)} : t \in [-t_1,t_2]\}$$

We have $\|u_0^{(n)}\|_{H^1} \le M$ for $n \gg 1$. We have $u^{(n)}$ bounded in

$$C^0([-T(M),T(M)],H^1) \cap C^1([-T(M),T(M)],H^{-1}).$$

Then there is a v in the above space with $u^{(n)}(t) \rightharpoonup v(t)$ for all in $t \in [-T(M),T(M)]$. By argument similar to those in Sec. 12.1 we have that v is a weak solution of (11.1) with $v(0) = u_0$. Hence by the uniqueness hypothesis we have $v = u$ and $[-T(M),T(M)] \subset (-S,T)$. Proceeding as for (12.18) we get $u^{(n)} \to u$ in $C^0([-T(M),T(M)],L^2)$ and proceeding like in (12.20) we have $u^{(n)} \to u$ in $C^0([-T(M),T(M)],L^{p+1})$. Furthermore since

$$\|u^{(n)}(t)\|_{H^1}^2 = 2E(u_0^{(n)}) + \|u_0^{(n)}\|_{L^2}^2 - \frac{2\lambda}{p+1}\int_{\mathbb{R}^d} |u^{(n)}(t)|^{p+1}dx,$$

we conclude that $\|u^{(n)}(t)\|_{H^1} \to \|u(t)\|_{H^1}$ in $C^0([-T(M),T(M)],\mathbb{R})$. This can be used, proceeding like in Claim 12.5,to get that $u^{(n)} \to u$ in $C^0([-T(M),T(M)],H^1)$. We can repeat the argument (replacing 0 with initial times t_0) and cover $[-t_1,t_2]$.

□

12.3 Uniqueness

To prove uniqueness we need preliminarily to have the following.

Lemma 12.8. *Let $u(t)$ be a weak solution of* (11.1). *Then*

$$u(t) = e^{\mathrm{i}t\triangle}u_0 - \mathrm{i}\lambda\int_0^t e^{\mathrm{i}(t-s)\triangle}|u(s)|^{p-1}u(s)ds. \tag{12.23}$$

Proof. Notice that $u(t)$ is a weak solution of the linear equation (10.2) with $f = \lambda|u|^{p-1}u$. If we apply Theorem 10.4 for $s = -1$ we conclude that formula

$$u(t) = e^{-\mathrm{i}t\triangle}u_0 - \mathrm{i}\int_0^t e^{-\mathrm{i}(t-t')\triangle}f(t')dt' \tag{10.9}$$

is true. This yields (12.23).

□

Proposition 12.9. *The solutions of* (12.23) *are unique.*

Proof. Let u and v be two solutions with same initial value. We have

$$u(t) - v(t) = \mathrm{i}\lambda \int_0^t e^{\mathrm{i}(t-t')\triangle}(|u(t')|^{p-1}u(t') - |v(t')|^{p-1}v(t'))dt'. \tag{12.24}$$

Then we apply Strichartz estimate to the admissible pair $(q, p+1)$

$$\begin{aligned} &\|u - v\|_{L^q([0,t],L^{p+1})} \le C\|(|u|^{p-1} + |v|^{p-1})(u - v)\|_{L^{q'}([0,t],L^{\frac{p+1}{p}})} \\ &\le C\||u| + |v|\|^{p-1}_{L^\infty([0,t],L^{p+1})}\|u - v\|_{L^{q'}([0,t],L^{p+1})} \le C\|u - v\|_{L^{q'}([0,t],L^{p+1})}. \end{aligned}$$

Now if $Ct^{\frac{1}{q'}-\frac{1}{q}} < 1$ (that is true for small times) and if all $t > 0$ we have $\|u-v\|_{L^q([0,t],L^{p+1})} > 0$, then

$$\|u - v\|_{L^{q'}([0,t],L^{p+1})} \le Ct^{\frac{1}{q'}-\frac{1}{q}}\|u - v\|_{L^q([0,t],L^{p+1})} < \|u - v\|_{L^q([0,t],L^{p+1})}.$$

This is absurd. It follows that for some $t > 0$ we have $\|u - v\|_{L^q([0,t],L^{p+1})} = 0$. From here we get uniqueness. □

We end this section with some easy remarks.

Corollary 12.10. *If $\lambda > 0$ the solutions are globally defined.*

Proof. Indeed we know by (11.7) and by the conservation of mass, that if a solution has maximal interval of existence $(-S, T)$ with $T < \infty$, we must have

$$\lim_{t\nearrow T} \|\nabla u(t)\|_{L^2} = +\infty \tag{12.25}$$

But for $\lambda > 0$ we have $\|\nabla u(t)\|_{L^2} \le 2E(u(t)) = 2E(u_0)$.

□

Corollary 12.11. *If $\lambda < 0$ and $1 < p < 1 + \frac{4}{d}$ the solutions are globally defined.*

Proof. We have

$$2E(u(t)) \ge \|\nabla u(t)\|^2_{L^2(\mathbb{R}^d)} - \frac{2|\lambda|}{p+1}C_p^{p+1}\|\nabla u(t)\|^{\alpha(p+1)}_{L^2(\mathbb{R}^d)}\|u_0\|^{(1-\alpha)(p+1)}_{L^2(\mathbb{R}^d)} \text{ for } \frac{1}{p+1} = \frac{1}{2} - \frac{\alpha}{d}.$$

Notice that

$$\alpha(p+1) = \frac{d}{2}(p+1) - d < 2 \Longleftrightarrow (p+1) - 2 < \frac{4}{d} \Longleftrightarrow p < 1 + \frac{4}{d}.$$

But then, if (12.25) happens, we have

$$\begin{aligned} 2E(u_0) &= \lim_{t\nearrow T} 2E(u(t)) \ge \lim_{t\nearrow T} \|\nabla u(t)\|^2_{L^2(\mathbb{R}^d)}\left(1 - \frac{2|\lambda|}{p+1}C_p^{p+1}\|\nabla u(t)\|^{\alpha(p+1)-2}_{L^2(\mathbb{R}^d)}\|u_0\|^{(1-\alpha)(p+1)}_{L^2(\mathbb{R}^d)}\right) \\ &= \lim_{t\nearrow T} \|\nabla u(t)\|^2_{L^2(\mathbb{R}^d)} = +\infty, \end{aligned}$$

which is absurd. $\square$

In the case $\lambda > 0$, it is known that for $p \geq 1 + \frac{4}{d}$, then for any solution there exist elements $u_\pm \in H^1(\mathbb{R}^d)$ s.t.

$$\lim_{t \pm \infty} \|u(t) - e^{-\mathrm{i}t\triangle} u_\pm\|_{H^1} = 0. \tag{12.26}$$

This is called completeness of the Wave Operators. For $p \leq 1 + \frac{2}{d}$ this is false, that is the asymptotic behavior of the nonlinear equation is much harder to understand. In the range $\lambda > 0$ and $p \in \left(1 + \frac{2}{d}, 1 + \frac{4}{d}\right)$ the existence of $u_\pm \in H^1(\mathbb{R}^d)$ s.t. (12.26) happens is an open problem. Instead, it is well known that if $u_0 \in H^1(\mathbb{R}^d)$ satisfies the additional condition that $|x|u_0 \in L^2(\mathbb{R}^d)$, then it is true that there exist $u_\pm \in H^1(\mathbb{R}^d)$ s.t. (12.26) happens.
The most interesting equations are those where $\lambda < 0$. We can take $\lambda = -1$. In this case there are solitary waves, that is solutions of the form

$$u(t,x) = e^{\frac{\mathrm{i}}{2} v \cdot x - \frac{i}{4}|v|^2 t + it\omega + i\gamma} \phi_\omega(x - vt - D)$$

where

$$-\Delta\phi_\omega + \omega\phi_\omega - |\phi_\omega|^{p-1}\phi_\omega = 0.$$

In 1–d there are explicit formulas

$$\begin{aligned} \phi(x) &= \frac{(\frac{p-1}{2} + 1)^{\frac{4}{p-1}}}{\cosh^{\frac{2}{p-1}}(\frac{p-1}{2}x)} \\ \phi_\omega(x) &= \omega^{\frac{1}{p-1}} \phi(\sqrt{\omega}x). \end{aligned} \tag{12.27}$$

For $d \geq 2$ there are many types of solitons. For example, the ones in (12.27) are *ground states*, and they are the only ones in $d = 1$. But in $d \geq 2$ there are also *excited states*. In general there are no multisolitons. However, the equation

$$\mathrm{i}u_t = -\partial_x^2 u - |u|^2 u \text{ for } (t,x) \in [0,\infty) \times \mathbb{R} \tag{12.28}$$

has multi–solitons, and is the famous cubic focusing NLS. It is remarkable because it is an *integrable system*. Specifically, there is a sort of nonlinear version of the Furier Transform, called Scattering Transform, that allows to diagonalize the equation (12.28). It is possible to construct very complex patterns of multi–soliton solutions, exploiting the scattering transform.

If we consider the L^2*–critical focusing NLS*

$$\mathrm{i}u_t = -\triangle u - |u|^{\frac{4}{d}} u \text{ in } \mathbb{R} \times \mathbb{R}^d,$$

from the discussion in Corollary 12.10, if C_d is the optimal constant for the Gagliardo Nirenberg inequality

$$\|u_0\|_{L^{2+\frac{4}{d}}(\mathbb{R}^d)}^{2+\frac{4}{d}} \leq C_d^{2+\frac{4}{d}} \|\nabla u_0\|_{L^2(\mathbb{R}^d)}^2 \|u_0\|_{L^2(\mathbb{R}^d)}^{\frac{4}{d}},$$

where we have computed

$$(1-\alpha)\left(2+\frac{4}{d}\right)=2+\frac{4}{d}-\frac{d}{2}\left(2+\frac{4}{d}\right)+d=2+\frac{4}{d}-d-2+d=\frac{4}{d}$$

it can be proved that for $\phi(x)$ the ground state satisfying

$$-\Delta\phi+\phi-|\phi|^{\frac{4}{d}}\phi=0$$

then we have

$$C_d=\left(\frac{2+\frac{4}{d}}{2\|\phi\|_{L^2}^{\frac{4}{d}}}\right)^{\frac{1}{2+\frac{4}{d}}}.$$

Now notice that if $\|u_0\|_{L^2}<\|\phi\|_{L^2}$ we have

$$2E(u(t))\geq\|\nabla u(t)\|^2_{L^2(\mathbb{R}^d)}\left(1-\frac{2}{2+\frac{4}{d}}\frac{2+\frac{4}{d}}{2\|\phi\|_{L^2}^{\frac{4}{d}}}\|u_0\|^{\frac{4}{d}}_{L^2(\mathbb{R}^d)}\right)=\|\nabla u(t)\|^2_{L^2(\mathbb{R}^d)}\left(1-\left(\frac{\|u_0\|_{L^2(\mathbb{R}^d)}}{\|\phi\|_{L^2(\mathbb{R}^d)}}\right)^{\frac{4}{d}}\right)$$

and we have global existence as in Corollary 12.11. □

13 Fujita's blow up theorem for semilinear heat equations

We will consider now a particular formulation of Fujita's classical blow up result. We consider the heat equation

$$\begin{cases} u_t=\triangle u+|u|^{p-1}u \text{ with } (t,x)\in(0,T)\times\mathbb{R}^d \\ u(0,x)=u_0(x) \text{ where } u_0\in C_0(\mathbb{R}^d,\mathbb{R}). \end{cases}$$

Here we recall that, like in (1.5),

$$C_0(\mathbb{R}^d,\mathbb{R}):=\{g\in C^0(\mathbb{R}^d,\mathbb{R}):\lim_{x\to\infty}g(x)=0\}.$$

We formulate this problem in the following integral form:

$$u(t)=e^{t\triangle}f+\int_0^t e^{(t-s)\triangle}|u(s)|^{p-1}u(s)ds. \tag{13.1}$$

It turns out that there exists a unique maximal solution of (13.13) with maximal lifespan T_f in $C^0([0,T_f,C_0(\mathbb{R}^d))$.
We will prove the following result.

Theorem 13.1. *Let $u_0\in C_0(\mathbb{R}^d)$ with $u_0\geq 0$ and $u_0\neq 0$ and suppose $1<p\leq 1+\frac{2}{d}$. Consider the solution of*

$$u(t)=e^{t\triangle}u_0+\int_0^t e^{(t-s)\triangle}u^p(s)ds \tag{13.2}$$

in $C^0([0,T_{u_0}),C_0(\mathbb{R}^d))$. Then $T_{u_0}<\infty$.

Remark 13.2. The original paper by Fujita [7] deals with the case $1 < p < 1 + \frac{2}{d}$. The proof we give is due to Weissler [15].

Somewhat related to Fujita's Theorem are theorems for dispersive equations, like the following, which is only a prototype of much more general results, and which we state only (for the proof see [12, p. 92]).

Theorem 13.3. *Let $u_1 \in C_c^2(\mathbb{R}^3, \mathbb{R})$ with $u_1 \geq 0$ and $u_1 \not\equiv 0$ and consider*

$$\begin{cases} (\partial_t^2 - \triangle)u - |u|^p = 0 \\ (u(0), \partial_t u(0)) = (0, u_1). \end{cases}$$

Then, if $1 < p < 1 + \sqrt{2}$ the solution blows up in finite time, in the sense that there exists a unique maximal solution $u \in C^2([0, T_{u_1}) \times \mathbb{R}^3, \mathbb{R})$ with $T_{u_1} < \infty$ where $u \notin L^\infty([0, T_{u_1}) \times \mathbb{R}^3)$.

□

13.1 Preliminaries on abstract dissipative semilinear equations

Definition 13.4 (Contraction semigroup)**.** Let X be a Banach space. A family $(S(t))_{t\geq 0} \in \mathcal{L}(X)$ is a contraction semigroup if the following happens.

(1) $\|S(t)\| \leq 1$ for all $t \geq 0$.

(2) $S(0) = I$.

(3) $S(t)S(s) = S(t+s)$ for all $t, s \geq 0$.

(4) For any $x \in X$ we have $S(t)x \in C^0([0, \infty), X)$.

Example 13.5. $S(t) := e^{t\triangle}$ is a contraction semigroup in $C_0(\mathbb{R}^d, \mathbb{R})$ (thought as a subspace of $L^\infty(\mathbb{R}^d, \mathbb{R})$). Indeed recall that for $K_t(x) := (4\pi t)^{-\frac{d}{2}} e^{-\frac{|x|^2}{4t}}$ we have $e^{t\triangle} f = K_t * f$ for all $f \in C_0(\mathbb{R}^d, \mathbb{R})$. Then $\|S(t)\| \leq \|S(t)1\|_\infty = 1$. We have $S(0) = I$. We have also $S(t+s)f = S(t)S(s)f$ for any $f \in C_c(\mathbb{R}^d, \mathbb{R})$, from

$$\begin{aligned} \mathcal{F}(K_{t+s} * f) &= e^{-t|\xi|^2} e^{-s|\xi|^2} \widehat{f} = (2\pi)^{-\frac{d}{2}} \mathcal{F}\left(\underbrace{\mathcal{F}^*(e^{-t|\xi|^2})}_{(2t)^{-\frac{d}{2}} e^{-\frac{|x|^2}{4t}}} * (K_s * f) \right) \\ &= \mathcal{F}\left(K_t * (K_s * f)\right) \Longrightarrow K_{t+s} * f = K_t * (K_s * f), \end{aligned}$$

and this extends to $f \in C_0(\mathbb{R}^d, \mathbb{R})$ by density. Finally, by Theorem 1.9 we have the continuity in $t = 0$ of $S(t)f$, and hence by (3) the continuity for all t.

Lemma 13.6. *Let $S(t)$ be a contraction semigroup, $F : X \to X$ a locally Lipschitz map, let $x \in X$ and let $u, v \in C^0([0, t_0], X)$ for $t_0 \in \mathbb{R}_+$ solve*

$$w(t) = S(t)x + \int_0^t S(t-s)F(w(s))ds. \tag{13.3}$$

Then $u = v$.

Let $M = \max_{0\le t\le t_0}\{\|u(t)\|, \|v(t)\|\}$. Then

$$\|u(t) - v(t)\| \le \int_0^t \|F(u(s)) - F(v(s))\|ds \le L(M) \int_0^t \|u(s) - v(s)\|ds$$

and apply Gronwall's inequality. □

Proposition 13.7. *Let $x \in X$ with $\|x\| \le M$. Then there is a unique solution $u \in C^0([0, T_M], X)$ of (13.3) with*

$$T_M := \frac{1}{2L(2M + \|F(0)\|) + 2}. \tag{13.4}$$

Proof. Set $K = 2M + \|F(0)\|$ and

$$E = \{u \in C^0([0, T_M], X) : \|u(t)\| \le K \text{ for all } t \in [0, T_M]\}$$

with the distance of $L^\infty([0, T_M], X)$. E is a complete metric space. Next consider the map $u \in E \to \Phi_u$

$$\Phi_u(t) = S(t)x + \int_0^t S(t-s)F(u(s))ds \text{ for all } t \in [0, T_M].$$

By $T_M = \frac{1}{2(L(K)+1)}$ for all $t \in [0, T_M]$ we have

$$\begin{aligned} &\|F(u(t))\| \le \|F(0)\| + \|F(u(t)) - F(0)\| \le \|F(0)\| + KL(K) \\ &= \|F(0)\| + (2M + \|F(0)\|)L(K) \le 2(M + \|F(0)\|)(L(K) + 1) = \frac{M + \|F(0)\|}{T_M} \end{aligned} \tag{13.5}$$

and

$$\|S(t)x\| \le \|x\| \le M. \tag{13.6}$$

So from (13.5)–(13.6) for $t \in [0, T_M]$ we have

$$\|\Phi_u(t)\| \le M + t\frac{M + \|F(0)\|}{T_M} \le 2M + \|F(0)\| = K$$

and so $\Phi_u \in E$.
For $u, v \in E$ we have

$$\|\Phi_u(t) - \Phi_v(t)\| \le \int_0^t \|F(u(s)) - F(v(s))\|ds \le T_M L(K)\|u - v\|_{L^\infty([0,T_M],X)}.$$

So by $T_M L(K) < 2^{-1}$

$$\|\Phi_u - \Phi_v\|_{L^\infty([0,T_M],X)} \le 2^{-1}\|u - v\|_{L^\infty([0,T_M],X)}$$

Hence $u \to \Phi_u$ is a contraction in E and so it has exactly one fixed point.

□

Notice that if $F(0) = 0$ if and $\lim_{M\to 0^+} L(M) = 0$, something which happens in many important cases, we can improve the above result and get a T_M s.t. $\lim_{M\to 0^+} T_M = \infty$, as we will see now.

Proposition 13.8. *Let $x \in X$ with $\|x\| \le M$. Assume $F(0) = 0$ Then there is a unique solution $u \in C^0([0,T_M],X)$ of (13.3) with*

$$T_M := \frac{1}{2L(2M)}. \tag{13.7}$$

Proof. The argument is the same. Here we set $K = 2M$ and define E as above by

$$E = \{u \in C^0([0,T_M],X) : \|u(t)\| \le 2M \text{ for all } t \in [0,T_M]\}$$

Consider the map $u \in E \to \Phi_u$ defined as above by

$$\Phi_u(t) = S(t)x + \int_0^t S(t-s)F(u(s))ds \text{ for all } t \in [0,T_M].$$

By $T_M = \frac{1}{2L(2M)}$ for all $t \in [0,T_M]$ we have

$$\|F(u(t))\| \le 2ML(2M) = \frac{M}{T_M} \tag{13.8}$$

and

$$\|T(t)x\| \le \|x\| \le M. \tag{13.9}$$

So from (13.5)–(13.6) for $t \in [0,T_M]$ we have

$$\|\Phi_u(t)\| \le M + t\frac{M}{T_M} \le 2M$$

and so $\Phi_u \in E$.
For $u, v \in E$ we have

$$\|\Phi_u(t) - \Phi_v(t)\| \le \int_0^t \|F(u(s)) - F(v(s))\|ds \le T_M L(2M)\|u - v\|_{L^\infty([0,T_M],X)}.$$

So by $T_M L(2M) = 2^{-1}$

$$\|\Phi_u - \Phi_v\|_{L^\infty([0,T_M],X)} \le 2^{-1}\|u - v\|_{L^\infty([0,T_M],X)}$$

Hence $u \to \Phi_u$ is a contraction in E and so it has exactly one fixed point. □

We now turn to an abstract form of the *maximum principle.*

Recall that in an ordered Banach space the ordering is characterized by a convex closed cone $\mathcal{C}$ s.t.

1. $\mathcal{C}+\mathcal{C}\subseteq\mathcal{C}$,

2. $\lambda\mathcal{C}\subseteq\mathcal{C}$ for all $\lambda\geq 0$ and

3. $\mathcal{C}\cap(-\mathcal{C})=\{0\}$.

Then given $x,y\in X$ we write $y\geq x$ if $(y-x)\in\mathcal{C}$.

Lemma 13.9. *Suppose that in X there is a relation of order and that $F(u)\geq 0$ if $u\geq 0$. Suppose furthermore that $S(t)$ is positivity preserving, that is $x\geq 0\Rightarrow S(t)x\geq 0$ for all t. Then if $x\geq 0$ the solution $u\in C^0([0,T_M],X)$ of Prop. 13.7 is $u(t)\geq 0$ for all t.*

Proof. We just rephrase the fixed point argument of Prop. 13.7 in a different set up. Indeed, if we redefine the set E writing

$$E=\{u\in C^0([0,T_M],X):\|u(t)\|\leq K \text{ and } u(t)\geq 0 \text{ for all } t\in[0,T_M]\},$$

then E is a complete metric space. Furthermore the map $u\to\Phi_u$ with

$$\Phi_u(t)=S(t)f+\int_0^t S(t-s)F(u(s))ds \text{ for all } t\in[0,T_M].$$

is such that $u(t)\geq 0$ for all $t\in[0,T_M]$ implies $\Phi_u(t)\geq 0$ for all $t\in[0,T_M]$. Then the proof of Proposition 13.7 works out in the same way as before under this slightly more restrictive definition of E.

□

Lemma 13.10. *Assume the hypotheses of Lemma 13.9 and furthermore that $F(v)\geq F(u)\geq 0$ if $v\geq u\geq 0$. Let $x<y$. Let $u(t),v(t)\in C^0([0,T_*),X)$ be solutions with $u(0)=x$ and $v(0)=y$. Then $u(t)\leq v(t)$ in $[0,T_*)$.*

Proof. If $M=\max\{\|x\|,\|y\|\}$, then using the setup of Prop. 13.7 we consider the set

$$E=\{f\in C^0([0,T_M],X):f(t)\geq 0 \text{ and } \|f(t)\|\leq K \text{ for all } t\in[0,T_M]\}$$

and the maps $f\in E\to\Phi_x(f)$ and $f\in E\to\Phi_y(f)$

$$\Phi_w(f)(t)=S(t)w+\int_0^t S(t-s)F(f(s))ds \text{ for all } t\in[0,T_M].$$

Let $v(t)$ be the solution with initial datum y. Then we have $\Phi_x(v)<\Phi_y(v)=v$. This can be iterated and if $0<\Phi_x^j(v)<\Phi_x^{j-1}(v)$, then $0<\Phi_x^{j+1}(v)<\Phi_x^j(v)$. But we know that $\Phi_x^j(v)\overset{j\to\infty}{\to}u$, with u the solution with initial datum x. Hence $u\leq v$.

So we have proved $u(t)\leq v(t)$ in $[0,T_M]$. Let now

$$T_1:=\sup\{T\in[0,T_*) \text{ such that } u(t)\leq v(t) \text{ in } [0,T]\}.$$

If $T_1 = T_*$ the theorem is finished. If $T_1 < T_*$ we have by continuity $u(T_1) \leq v(T_1)$. But then there exists a $0 < T < T_* - T_1$ with s.t. $\widetilde{u}(t) := u(t + T_1)$ and resp. $\widetilde{v}(t) := v(t + T_1)$ solve in $[0, T]$ the equation with initial data $\widetilde{x} \leq \widetilde{y}$ with $\widetilde{x} := u(T_1)$ and resp. $\widetilde{y} := v(T_1)$. But for T small enough we have $\widetilde{u}(t) \leq \widetilde{v}(t)$ in $[0, T]$ by the 1st part of the proof. But this implies than $u(t) \leq v(t)$ in $[0, T_1 + T]$. This is absurd by the definition of T_1, and so $T_1 = T_*$.

□

We will consider now the function $T : X \to (0, \infty]$ where for any $x \in X$ the interval $[0, T(x))$ is the maximal (positive) interval of existence of the unique solution of (13.3).

Theorem 13.11. *We have, for $u(t)$ the corresponding solution in $C([0, T(x)), X)$,*

$$2L(\|F(0)\| + 2\|u(t)\|) \geq \frac{1}{T(x) - t} - 2 \tag{13.10}$$

for all $t \in [0, T(x))$. We have the alternatives

(1) $T(x) = +\infty$;

(2) if $T(x) < +\infty$ then $\lim_{t \nearrow T(x)} \|u(t)\| = +\infty$.

Proof. First of all it is obvious that if $T(x) < +\infty$ then by (13.10)

$$\lim_{t \nearrow T(x)} L(\|F(0)\| + 2\|u(t)\|) = +\infty \Rightarrow \lim_{t \nearrow T(x)} \|u(t)\| = +\infty$$

where the implication follows from the fact that $M \to L(M)$ is an increasing function.

Let $x \in X$. Set $T(x) = \sup\{T > 0 : \exists u \in C^0([0, T), X) \text{ solution of (13.3) }\}$. We are left with the proof of (13.10), which is clearly true if $T(x) = \infty$. Now suppose that $T(x) < \infty$ and that (13.10) is false. This means that there exists a $t \in [0, T(x))$ with

$$\frac{1}{T_M} - 2 = 2L(\|F(0)\| + 2\|u(t)\|) < \frac{1}{T(x) - t} - 2 \Rightarrow T(x) - t < T_M$$

for $M = \|u(t)\|$, where we recall $T_M := \frac{1}{2L(2M + \|F(0)\|) + 2}$ in (13.4). Consider now $v \in C^0([0, T_M], X)$ the solution of

$$v(s) = S(s)u(t) + \int_0^s S(s - s')F(v(s'))ds' \text{ for all } s \in [0, T_M].$$

which exists by the previous Proposition 13.7. Then define

$$w(s) := \begin{cases} u(s) \text{ for } s \in [0, t] \\ v(s - t) \text{ for } s \in [t, T_M]. \end{cases}$$

We claim that $w \in C^0([0,T_M],X)$ is a solution of (13.3). In $[0,t]$ this is obvious since in $w = u$ in $[0,t]$ and $u \in C^0([0,t],X)$ is a solution of (13.3). Let now $s \in (t,T_M]$. We have

$$
\begin{aligned}
w(s) = v(s-t) &= S(s-t)u(t) + \int_0^{s-t} S(s-t-s')F(v(s'))ds' \\
&= S(s-t)\left[S(t)x + \int_0^t S(t-s')F(u(s'))ds'\right] + \int_0^{s-t} S(s-t-s')F(v(s'))ds' \\
&= S(s)x + \int_0^t S(s-s')F(\underbrace{u(s')}_{w(s')})ds' + \int_t^s S(s-s')F(\underbrace{v(s'-t)}_{w(s')})ds' \\
&= S(s)x + \int_0^s S(s-s')F(w(s'))ds.
\end{aligned}
$$

□

Remark 13.12. Notice that if $F(0) = 0$, then we can prove the improved estimate

$$2L(\|F(0)\| + 2\|u(t)\|) \geq \frac{1}{T(x)-t}. \tag{13.11}$$

The proof is exactly the same of Theorem 13.11 using the altered definitions of T_M, $T_M = (2L(2M))^{-1}$.

Proposition 13.13. *(1) $T : X \to (0,\infty]$ is lower semicontinuous;*

(2) if $x_n \to x$ in X and if $\bar{t} < T(x)$ we have $u_n \to u$ in $C^0([0,\bar{t}],X)$ with u_n the solution of (13.3) *with initial datum x_n.*

Proof. Let $u \in C^0([0,T(x)),X)$ be the solution of (13.3) and consider $\bar{t} < T(x)$. Set $M = 2\|u\|_{L^\infty([0,\bar{t}],X)}$ and let

$$\tau_n = \sup\{t \in [0,T(x_n)) : \|u_n\|_{L^\infty([0,t],X)} \leq K\} \text{ where } K = 2M + \|F(0)\|.$$

For $n \gg 1$ we have $\|x_n\| < M$. Then $u_n \in C^0([0,T_M],X)$ with $\|u_n\|_{L^\infty([0,T_M],X)} \leq K$ by Prop. 13.7. This implies $\tau_n \geq T_M$. For $0 \leq t \leq \min\{\bar{t},\tau_n\}$ we have

$$u(t) - u_n(t) = S(t)(x-x_n) + \int_0^t S(s-t)(F(u(s)) - F(u_n(s)))ds$$

and so

$$
\begin{aligned}
&\|u(t) - u_n(t)\| \leq \|x - x_n\| + L(K)\int_0^t \|u(s) - u_n(s)\|ds \Rightarrow \\
&\|u(t) - u_n(t)\| \leq e^{L(K)t}\|x - x_n\| \Rightarrow \|u(t) - u_n(t)\| \leq e^{L(K)\bar{t}}\|x - x_n\|.
\end{aligned} \tag{13.12}
$$

So $\|u_n(t)\| \leq \|u(t)\| + e^{L(K)\bar{t}}\|x - x_n\| \leq M/2 + e^{L(K)\bar{t}}\|x - x_n\| \leq M$ for $n \gg 1$ and $0 \leq t \leq \min\{\bar{t},\tau_n\}$. This and continuity imply $\tau_n > \min\{\bar{t},\tau_n\}$ and so $\tau_n > \bar{t}$. Then we have $T(x_n) > \bar{t}$. This implies the lower semi–continuity in claim (1). Furthermore by (13.12) we have also $u_n \to u$ in $C^0([0,\bar{t}],X)$. □

13.2 Proof of Fujita's Theorem

We know that $S(t) := e^{t\triangle}$ is a contraction semigroup in $C_0(\mathbb{R}^d, \mathbb{R})$. Notice that in $C_0(\mathbb{R}^d, \mathbb{R})$ there is a natural partial order, and that this is preserved by $e^{t\triangle}$. In fact, if $f \in C_0(\mathbb{R}^d, \mathbb{R})$ is $f(x) \geq 0$ for all $x \in \mathbb{R}^d$, and is not identically 0, then $e^{t\triangle} f > 0$ everywhere ($e^{t\triangle}$ is *positivity enhancing*).

By the abstract theory presented above, we can prove the following maximum principle property.

Lemma 13.14. *Let $u \in C([0,T), C_0(\mathbb{R}^d, \mathbb{R}))$ be the unique maximal solution of*

$$u(t) = e^{t\triangle} f + \int_0^t e^{(t-s)\triangle} |u(s)|^{p-1} u(s) ds \tag{13.13}$$

and let $f \geq 0$. Then $u(t,x) \geq 0$ for all $(t,x) \in [0,T) \times \mathbb{R}^n$.

□

We prove now the following version of Fujita's Theorem (compared to Theorem 13.1, we add the hypothesis $u_0 \in L^1(\mathbb{R}^d)$).

Theorem 13.15. *Let $u_0 \in L^1(\mathbb{R}^d) \cap C_0(\mathbb{R}^d)$ with $u_0 \geq 0$ and suppose $1 < p \leq 1 + \frac{2}{d}$. Suppose that $u(t) \in C^0([0, T_{u_0}), C_0(\mathbb{R}^d))$ is a positive solution of*

$$u(t) = e^{t\triangle} u_0 + \int_0^t e^{(t-s)\triangle} u^p(s) ds. \tag{13.14}$$

Then $T_{u_0} < \infty$.

Proof. We claim, and for the moment assume, the following inequality due to Weissler:

$$t^{\frac{1}{p-1}} e^{t\triangle} u_0(x) \leq C \text{ for a fixed } C = C(p) > 0, \text{ for any } x \in \mathbb{R}^d,\ t \in [0, T_{u_0}) \text{ and any } u_0 \geq 0. \tag{13.15}$$

Here, crucially, C depends only on p.
Suppose we have $T_{u_0} = \infty$ and assume (13.15).
By dominated convergence we have for any $x \in \mathbb{R}^d$

$$\lim_{t \nearrow \infty} (4\pi)^{\frac{d}{2}} t^{\frac{d}{2}} e^{t\triangle} u_0(x) = \lim_{t \nearrow \infty} \int_{\mathbb{R}^d} e^{-\frac{|x-y|^2}{4t}} u_0(y) dy = \int_{\mathbb{R}^n} u_0(y) dy = \|u_0\|_{L^1(\mathbb{R}^n)}. \tag{13.16}$$

In the particular case $p < 1 + \frac{2}{d}$, equivalent to $\frac{1}{p-1} - \frac{d}{2} > 0$, we see immediately that (13.16) is incompatible with (13.15) since

$$\lim_{t \nearrow \infty} t^{\frac{1}{p-1}} e^{t\triangle} u_0(x) = \lim_{t \nearrow \infty} t^{\frac{1}{p-1} - \frac{d}{2}} t^{\frac{d}{2}} e^{t\triangle} u_0(x) = \lim_{t \nearrow \infty} t^{\frac{1}{p-1} - \frac{d}{2}} (4\pi)^{-\frac{d}{2}} \|u_0\|_{L^1(\mathbb{R}^n)} = +\infty.$$

In the case $p = 1 + \frac{2}{d}$ this argument does not provide a contradiction for all u_0 (although this argument shows that if $\|u_0\|_{L^1(\mathbb{R}^d)} > (4\pi)^{\frac{d}{2}} C$ for $C = C(1 + \frac{2}{d})$ then there is blow up). We complete the argument below, but first we prove claim (13.15).

Proof of (13.15) We turn now to the proof of (13.15). We have $u(t) \geq e^{t\triangle}u_0(x)$ and

$$\begin{aligned} u(t) &\geq \int_0^t e^{(t-s)\triangle}u^p(s)ds \geq \int_0^t e^{(t-s)\triangle}(e^{s\triangle}u_0)^p ds \\ &\geq \int_0^t (e^{(t-s)\triangle}e^{s\triangle}u_0)^p ds = \int_0^t (e^{t\triangle}u_0)^p ds = t(e^{t\triangle}u_0)^p, \end{aligned} \tag{13.17}$$

where we used, for $d\mu(y) := (4\pi\tau)^{-\frac{d}{2}}e^{-\frac{|x-y|^2}{4\tau}}dy$ which gives a probability measure in $\mathbb{R}^d$,

$$\begin{aligned} e^{\tau\triangle}(f)^p(x) &= (4\pi\tau)^{-\frac{d}{2}}\int_{\mathbb{R}^d} e^{-\frac{|x-y|^2}{4\tau}}f^p(y)dy = \int_{\mathbb{R}^d} f^p(y)d\mu(y) \\ &\geq \left(\int_{\mathbb{R}^d} f(y)d\mu(y)\right)^p = \left((4\pi\tau)^{-\frac{d}{2}}\int_{\mathbb{R}^d} e^{-\frac{|x-y|^2}{4\tau}}f(y)dy\right)^p = \left(e^{\tau\triangle}(f)(x)\right)^p, \end{aligned}$$

which follows from Jensen's inequality $\varphi(\int f d\mu) \leq \int \varphi \circ f d\mu$ for a convex function φ and a probability measure μ.

By a substitution inside (13.17) and by repeating the same argument we get

$$u(t) \geq \int_0^t e^{(t-s)\triangle}s^p(e^{s\triangle}u_0)^{p^2}ds \geq \int_0^t s^p(e^{t\triangle}u_0)^{p^2}ds = \frac{t^{p+1}}{p+1}(e^{t\triangle}u_0)^{p^2}.$$

This is the case $k=2$ of the following inequality which for any $k \in \mathbb{N}$ with $k \geq 2$ we will obtain by induction:

$$u(t) \geq \frac{t^{1+p+...+p^{k-1}}(e^{t\triangle}u_0)^{p^k}}{(1+p)^{p^{k-2}}(1+p+p^2)^{p^{k-3}}...(1+p+...+p^{k-1})} = \frac{t^{\frac{p^k-1}{p-1}}(e^{t\triangle}u_0)^{p^k}}{\prod_{\ell=2}^k\left(\frac{p^\ell-1}{p-1}\right)^{p^{k-\ell}}}. \tag{13.18}$$

Indeed, assuming (13.18) for k and repeating (13.17) we have

$$\begin{aligned} u(t) &\geq \int_0^t e^{(t-s)\triangle}u^p(s)ds \geq \int_0^t \frac{s^{\frac{p^k-1}{p-1}p}}{\prod_{\ell=2}^k\left(\frac{p^\ell-1}{p-1}\right)^{p^{k+1-\ell}}}e^{(t-s)\triangle}(e^{s\triangle}u_0)^{p^{k+1}}ds \\ &\geq \int_0^t \frac{s^{\frac{p^k-1}{p-1}p}}{\prod_{\ell=2}^k\left(\frac{p^\ell-1}{p-1}\right)^{p^{k+1-\ell}}}ds(e^{t\triangle}u_0)^{p^{k+1}} = \frac{t^{\frac{p^k-1}{p-1}p+1}}{\prod_{\ell=2}^k\left(\frac{p^\ell-1}{p-1}\right)^{p^{k+1-\ell}}\left(\frac{p^k-1}{p-1}p+1\right)}(e^{t\triangle}u_0)^{p^{k+1}} \\ &= \frac{t^{\frac{p^{k+1}-1}{p-1}}}{\prod_{\ell=2}^k\left(\frac{p^\ell-1}{p-1}\right)^{p^{k+1-\ell}}\frac{p^{k+1}-1}{p-1}}(e^{t\triangle}u_0)^{p^{k+1}} = \frac{t^{\frac{p^{k+1}-1}{p-1}}}{\prod_{\ell=2}^{k+1}\left(\frac{p^\ell-1}{p-1}\right)^{p^{k+1-\ell}}}(e^{t\triangle}u_0)^{p^{k+1}}. \end{aligned}$$

So (13.18) holds also for $k+1$ and hence for any $k \in \mathbb{N}$ with $k \geq 2$. Then

$$\begin{aligned} t^{\frac{p^k-1}{(p-1)p^k}}e^{t\triangle}u_0 &\leq (u(t))^{\frac{1}{p^k}}\prod_{\ell=2}^k\left(\frac{p^\ell-1}{p-1}\right)^{\frac{1}{p^\ell}} \Rightarrow t^{\frac{1}{p-1}}e^{t\triangle}u_0 \leq \prod_{\ell=2}^\infty\left(\frac{p^\ell-1}{p-1}\right)^{\frac{1}{p^\ell}} \\ &= e^{\sum_{\ell=2}^\infty p^{-\ell}\log\left(\frac{p^\ell-1}{p-1}\right)} = e^{\sum_{\ell=2}^\infty p^{-\ell}\log\left(\sum_{j=1}^{\ell-1}p^j\right)} \leq e^{\sum_{\ell=2}^\infty p^{-\ell}\log\left(\ell p^\ell\right)} < +\infty. \end{aligned}$$

This proves (13.15).

Proof of the case $p = 1 + \frac{2}{d}$ We return to the proof of Theorem 13.15 when $p = 1 + \frac{2}{d}$. If instead of looking at solutions in $C_0(\mathbb{R}^d)$ we look at solutions in $X := C_0(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$ then our $u \in C^0([0, T_{u_0}), C_0(\mathbb{R}^d))$ is also $u \in C^0([0, T_{u_0}), X)$. Indeed, if the lifespan in X was shorter, then for some $t_0 < T_{u_0}$ we would have

$$\lim_{t \nearrow t_0} \|u(t)\|_{L^1(\mathbb{R}^d)} = \infty \text{ while } \sup_{0 \le t \le t_0} \|u(t)\|_{L^\infty(\mathbb{R}^d)} < \infty.$$

But this is impossible because from (13.14) for $t < t_0$ we get

$$\|u(t)\|_{L^1(\mathbb{R}^d)} \le \|u_0\|_{L^1(\mathbb{R}^d)} + \int_0^t \|u(s)\|^{p-1}_{L^\infty(\mathbb{R}^d)} \|u(s)\|_{L^1(\mathbb{R}^d)} ds$$

implies by the Gronwall inequality

$$\|u(t)\|_{L^1(\mathbb{R}^d)} \le \|u_0\|_{L^1(\mathbb{R}^d)} e^{t_0 (\sup_{0 \le t \le t_0} \|u(t)\|_{L^\infty(\mathbb{R}^d)})^{p-1}} < \infty$$

and so

$$+\infty = \lim_{t \nearrow t_0} \|u(t)\|_{L^1(\mathbb{R}^d)} \le \|u_0\|_{L^1(\mathbb{R}^d)} e^{t_0 (\sup_{0 \le t \le t_0} \|u(t)\|_{L^\infty(\mathbb{R}^d)})^{p-1}} < +\infty,$$

which is absurd.
Hence we conclude that $t_0 = T_{u_0}$ and we have $u \in C^0([0, T_{u_0}), L^1(\mathbb{R}^d))$, and so $u(t) \in L^1(\mathbb{R}^d)$ for all $t \in [0, T_{u_0})$. Since any such t can be taken as an initial value at time t for our solution, it follows that

$$\tau^{\frac{d}{2}} (e^{\tau \triangle} u(t))(x) \le C \text{ for a fixed } C > 0 \text{ , any } x \in \mathbb{R}^d \text{ and } 0 < \tau < T_{u_0} - t$$

and for all $t \in [0, T_{u_0})$. In particular if $T_{u_0} = \infty$, by the argument in (13.16), we have

$$\|u(t)\|_{L^1(\mathbb{R}^d)} \le (4\pi)^{\frac{d}{2}} C \text{ for all } t \ge 0. \tag{13.19}$$

Initially we assume that $u_0 \ge k K_\alpha$, for $K_\alpha(x) := (4\pi\alpha)^{-\frac{d}{2}} e^{-\frac{|x|^2}{4\alpha}}$. Notice that $K_\alpha = e^{\alpha \triangle} \delta_0$. Then we have (a bit formally, but can be checked)

$$u(t) \ge e^{t\triangle} u_0 \ge k e^{t\triangle} K_\alpha = k e^{t\triangle} e^{\alpha\triangle} \delta_0 = k e^{(\alpha+t)\triangle} \delta_0 = k K_{\alpha+t}.$$

Now we have

$$\|u(t)\|_{L^1(\mathbb{R}^d)} \geq \| \int_0^t e^{(t-s)\triangle} u^p(s) ds\|_{L^1(\mathbb{R}^d)} = \int_{\mathbb{R}^d} dx \int_0^t e^{(t-s)\triangle} u^p(s)(x) ds$$

$$= \int_0^t ds \int_{\mathbb{R}^d} dx e^{(t-s)\triangle} u^p(s)(x) = \int_0^t \|e^{(t-s)\triangle} u^p(s)\|_{L^1(\mathbb{R}^d)} ds \text{ (by commuting the order of integration)}$$

$$\geq \int_0^t \|e^{(t-s)\triangle}(e^{s\triangle}u_0)^p\|_{L^1(\mathbb{R}^d)} ds$$

$$= \int_0^t ds \int_{\mathbb{R}^d} dx \int_{\mathbb{R}^n} dy K_{t-s}(x-y)(e^{s\triangle}u_0)^p(y) = \int_0^t ds \int_{\mathbb{R}^d} dy (e^{s\triangle}u_0)^p(y) \underbrace{\int_{\mathbb{R}^d} dx K_{t-s}(x-y)}_{1}$$

$$= \int_0^t \|(e^{s\triangle}u_0)^p\|_{L^1(\mathbb{R}^d)} ds \geq k^p \int_0^t \|(e^{s\triangle}K_\alpha)^p ds\|_{L^1(\mathbb{R}^d)} = k^p \int_0^t \|K^p_{\alpha+s}\|_{L^1(\mathbb{R}^d)} ds.$$

Now notice that

$$K^p_\beta(x) = (4\pi\beta)^{-\frac{d}{2}p} e^{-\frac{p|x|^2}{4\beta}} = (4\pi\beta)^{-\frac{d}{2}(p-1)} p^{-\frac{d}{2}} (4\pi\beta/p)^{-\frac{d}{2}} e^{-\frac{p|x|^2}{4\beta}} \quad = (4\pi\beta)^{-\frac{d}{2}(p-1)} p^{-\frac{d}{2}} K^p_{\frac{\beta}{p}}(x)$$

$$= (4\pi\beta)^{-1} p^{-\frac{d}{2}} K^p_{\frac{\beta}{p}}(x) \qquad \text{by } p = 1 + 2/d.$$

This implies that, if by contradiction we suppose $T_{u_0} = +\infty$, then we have

$$\|u(t)\|_{L^1(\mathbb{R}^d)} \geq p^{-\frac{d}{2}} k^p \int_0^t (4\pi(\alpha+s))^{-1} \|K_{\frac{\alpha+s}{p}}\|_{L^1(\mathbb{R}^d)} ds$$

$$= p^{-\frac{d}{2}} k^p (4\pi)^{-1} \int_0^t (\alpha+s)^{-1} ds \to +\infty \text{ as } t \nearrow \infty.$$

This contradicts (13.19).
Suppose now we don't have $u_0 \geq kK_\alpha$. Let us set $v(t) = u(t+\varepsilon)$ for some $\varepsilon > 0$. Then $v(t)$ is a solution of (13.14) with initial value $u(\varepsilon)$. We have $u(\varepsilon) \geq e^{\varepsilon\triangle}u_0$

$$v(0) = u(\varepsilon) \geq e^{\varepsilon\triangle}u_0 = (4\pi\varepsilon)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{-\frac{|x-y|^2}{4\varepsilon}} f(y) dy = (4\pi\varepsilon)^{-\frac{d}{2}} e^{-\frac{|x|^2}{2\varepsilon}} \int_{\mathbb{R}^d} e^{\frac{|x+y|^2}{4\varepsilon}} e^{-\frac{|y|^2}{2\varepsilon}} f(y) dy$$

$$\geq (4\pi\varepsilon)^{-\frac{d}{2}} e^{-\frac{|x|^2}{2\varepsilon}} \int_{\mathbb{R}^d} e^{-\frac{|y|^2}{2\varepsilon}} f(y) dy = kK_{\frac{\varepsilon}{2}}$$

where we used the parallelogram formula

$$|x+y|^2 + |x-y|^2 = 2|x|^2 + 2|y|^2.$$

But then $v(t)$ blows up in finite time, and so $u(t)$ does too. This completes the proof of Theorem 13.15 also in the case $p = 1 + \frac{2}{d}$.

□

So far we have proved the blow up when $1 < p \leq 1 + \frac{2}{d}$ for positive initial data with $u_0 \in C^0_0(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$. But in fact the result holds for $u_0 \in C^0_0(\mathbb{R}^d)$ because of the maximum principle.

Lemma 13.16. *Suppose that* $0 \le v_0 \le u_0$ *are in* $C_0^0(\mathbb{R}^d)$ *and let* $u(t), v(t) \in C^0([0,T], C_0^0(\mathbb{R}^d))$ *be corresponding solutions of* (13.14). *Then* $u(t) \ge v(t)$.

This follows by Lemma 13.10 and means that if $u_0 \in C_0^0(\mathbb{R}^d)$ but $u_0 \notin L^1(\mathbb{R}^d)$, the solution u blows up, because we can find a $0 \le v_0 \le u_0$ with $v_0 \in C_0^0(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$ and v_0 non zero whose corresponding $v(t)$ blows up. Then by the maximum principle also $u(t)$ blows up. □

This completes the proof of Theorem 13.1. □

Remark 13.17. The coefficient $p = 1 + \frac{2}{d}$ is critical. In fact, for any $p > 1 + \frac{2}{p}$ there exists $\epsilon_p > 0$ s.t. if $u_0 \in X := C_0^0(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$ satisfies $\|u_0\|_X < \epsilon_p$, then equation (13.14) admits a global solution in $C_b^0([0,\infty), C_0^0(\mathbb{R}^d) \cap L^1(\mathbb{R}^d))$.

A Appendix. On the Bochner integral

For this part see [3]. Let X be a Banach space.

Definition A.1 (Strong measurability)**.** Let I be an interval. A function $f : I \to X$ is strongly measurable if there exists a set E of measure 0 and a sequence $(f_n(t))$ in $C_c(I, X)$ s.t. $f_n(t) \to f(t)$ for any $t \in I \backslash E$.

Remark A.2. Notice that when $\dim X < \infty$ a function is measurable (in the sense that $f^{-1}(B)$ is measurable for any Borel set B) if an only if it is strongly measurable in the above sense. Indeed if f is strongly measurable in the above sense then as a point wise limit of measurable functions f is measurable, see Theorem 1.14 p. 14 Rudin [11]. Viceversa if f is measurable, then f is strongly measurable in the above sense, see the Corollary to Lusin's Theorem in Rudin [11] p. 54.

Example A.3. Consider $\{x_j\}_{j=1}^n$ in X and $\{A_j\}_{j=1}^n$ measurable sets in I with $|A_j| < \infty$ and with $A_j \cap A_k = \emptyset$ for $j \ne k$. Then we claim that the *simple* function

$$f(t) := \sum_{j=1}^n x_j \chi_{A_j}(t) : I \to X \tag{A.1}$$

is measurable. Indeed, see Rudin [11] p. 54, there are sequences $\{\varphi_{j,k}\}_{k\in\mathbb{N}}$ in $C_c^0(I, \mathbb{R})$ with $\varphi_{j,k}(t) \stackrel{k\to\infty}{\to} \chi_{A_j}(t)$ a.e. and hence

$$C_c^0(I, \mathbb{R}) \ni f_k(t) := \sum_{j=1}^n x_j \varphi_{j,k}(t) \stackrel{k\to\infty}{\to} f(t) \text{ a.e. in } I.$$

Proposition A.4. *If* (f_n) *is a sequence of strongly measurable functions from* I *to* X *convergent a.e. to a* $f : I \to X$, *then* f *is strongly measurable.*

Proof. There is an E with $|E| = 0$ s.t. $f_n(t) \overset{n\to\infty}{\to} f(t)$ for any $t \in I\backslash E$. Consider for any n a sequence $C_c(I, X) \ni f_{n,k} \overset{k\to\infty}{\to} f_n$ a.e. We will suppose first that $|I| < \infty$. By applying Egorov Theorem to $\{\|f_{n,k} - f_n\|\}_{k\in\mathbb{N}}$ there is $E_n \subset I$ with $|E_n| \le 2^{-n}$ s.t. $\|f_{n,k} - f_n\| \overset{k\to\infty}{\to} 0$ uniformly in $I\backslash E_n$ Let $k(n)$ be s.t. $\|f_{n,k(n)} - f_n\| < 1/n$ in $I\backslash E_n$ and set $g_n = f_{n,k(n)}$. Set $F := E \bigcup (\bigcap_m \bigcup_{n>m} E_n)$. Then $|F| = 0$. Indeed for any m

$$|F| \le |E| + \sum_{n=m}^{\infty} |E_n| \le |E| + \sum_{n=m}^{\infty} 2^{-n} \overset{m\to\infty}{\to} 0.$$

Let $t \in I\backslash F$. Since $t \notin E$ we have $f_n(t) \overset{n\to\infty}{\to} f(t)$. Furthermore, for n large enough we have $t \in I\backslash E_n$. Indeed

$$t \notin \bigcap_m \bigcup_{n>m} E_n \Rightarrow \exists\ m \text{ s.t. } t \notin \bigcup_{n>m} E_n \Rightarrow\ t \notin E_n\ \forall\ n > m.$$

Then $\|g_n(t) - f_n(t)\| < 1/n$ and $g_n(t) \overset{n\to\infty}{\to} f(t)$. So $f(t)$ is measurable in the case $|I| < \infty$. Now we consider the case $|I| = \infty$. We express $I = \cup_n I_n$ for an increasing sequence of intervals with $|I_n| < \infty$. Consider for any n a sequence $C_c(I_n, X) \ni f_{n,k} \overset{k\to\infty}{\to} f$ a.e. in I_n. Then by Egorov Theorem to $\|f_{n,k} - f_n\|$ there is $E_n \subset I_n$ with $|E_n| \le 2^{-n}$ s.t. $f_{n,k} \overset{k\to\infty}{\to} f_n$ uniformly in $I_n\backslash E_n$ Let $k(n)$ be s.t. $\|f_{n,k(n)} - f_n\| < 1/n$ in $I_n\backslash E_n$ and set $g_n = f_{n,k(n)}$. Then defining F like above, the remainder of the proof works exactly like for the case $|I| < \infty$. □

Example A.5. Consider a sequence $\{x_j\}_{j\in\mathbb{N}}$ in X and a sequence $\{A_j\}_{j\in\mathbb{N}}$ of measurable sets in I with $|A_j| < \infty$ and with $A_j \cap A_k = \emptyset$ for $j \ne k$. Then we claim

$$f(t) := \sum_{j=1}^{\infty} x_j \chi_{A_j}(t) \tag{A.2}$$

is measurable. Indeed if we set $f_n(t) := \sum_{j=1}^{n} x_j \chi_{A_j}(t)$, then we have $\lim_{n\to\infty} f_n(t) = f(t)$ for any t, since if $t \notin \cup_{j=1}^{\infty} A_j$ both sides are 0, and if $t \in A_{n_0}$ then for $n \ge n_0$ we have $f_n(t) = x_{n_0} = f(t)$. Hence by Proposition A.4 the function f is measurable.
When the sum in (A.2) is finite then the function f is called *simple.*

Example A.6. Consider a sequence $\{x_j\}_{j\in\mathbb{N}}$ in X and a sequence $\{A_j\}_{j\in\mathbb{N}}$ of measurable sets in I where again $A_j \cap A_k = \emptyset$ for $j \ne k$ but we allow $|A_j| = \infty$. Then

$$f(t) := \sum_{j=1}^{\infty} x_j \chi_{A_j}(t) \tag{A.3}$$

is measurable. To see this consider $f_n(t) = \chi_{[-n,n]}(t) f(t)$. Then

$$f_n(t) = \sum_{j=1}^{\infty} x_j \chi_{A_j \cap [-n,n]}(t)$$

and by Example A.5 we know that each $f_n(t)$ is strongly measurable. Since $f_n(t) \to f(t)$ for any $t \in I$ we conclude by Proposition A.4 that f is strongly measurable.

Another natural definition of measurability is the following one.

Definition A.7 (Weak measurability)**.** Let I be an interval. A function $f : I \to X$ is weakly measurable if for any $x' \in X'$ the function $t \to \langle x', f(t)\rangle_{X'X}$ is a measurable function $I \to \mathbb{R}$.

Obviously, strongly measurable implies weakly measurable. Let us explore the viceversa.

Definition A.8. Let I be an interval. A function $f : I \to X$ is *almost separably valuable* if there exists a 0 measure set $N \subset I$ s.t. $f(I\backslash N)$ is separable.

The following lemma shows that strongly measurable functions are almost separably valuable.

Lemma A.9. *If $f : I \to X$ is strongly measurable with $(f_n(t))$ a sequence in $C_c(I, X)$ s.t. $f_n(t) \to f(t)$ for any $t \in I\backslash E$ for a 0 measure set $E \subset I$ then $f(I\backslash E)$ is separable and there exists a separable Banach subspace $Y \subseteq X$ with $f(I\backslash E) \subseteq Y$.*

Proof. First of all $f_n(I \cap \mathbb{Q})$ is a countable dense set in $f_n(I)$. So $f_n(I)$ is separable. In a separable metric space any subspace is separable. So $f_n(I\backslash E)$ is separable. The closed vector space Y generated by $\cup_n f_n(I\backslash E)$ is separable. Indeed let $C \subseteq \cup_n f_n(I\backslash E)$ be a countable set dense in $\cup_n f_n(I\backslash E)$. Let $\mathrm{Span}_{\mathbb{Q}}(C)$ be the vector space on $\mathbb{Q}$ generated by C. Then $\mathrm{Span}_{\mathbb{Q}}(C)$ is dense in Y. For $C = \{x_1, x_2, ...\}$ we have $\mathrm{Span}_{\mathbb{Q}}(C) = \cup_{n=1}^{\infty}\mathrm{Span}_{\mathbb{Q}}(\{x_1, ..., x_n\})$. This proves that $\mathrm{Span}_{\mathbb{Q}}(C)$ is countable and that Y is separable. □

Example A.10. Let X be a Hilbert space with an orthonormal basis $\{e_t\}_{t\in\mathbb{R}}$. Then the map $f : \mathbb{R} \to X$ given by $f(t) = e_t$ is not strongly measurable. This follows from the fact that it is not almost separably valuable.
On the other hand if $x \in X$ then $t \to \langle f(t), x\rangle$ is different from 0 only on a countable subset of $\mathbb{R}$, and as such it is measurable. Hence f is weakly measurable.
Notice however that if $C \subset [0, 1]$ is the standard Cantor set (which has 0 measure and has same cardinality of $\mathbb{R}$) and if $\{\widetilde{e}_t\}_{t\in C}$ is another basis of X, then the map

$$g(t) = \begin{cases} \widetilde{e}_t \text{ for } t \in C \text{ and} \\ \quad 0 \text{ otherwise} \end{cases}$$

is weakly measurable (like f and for the same reasons) and is almost separably valuable. Pettis Theorem, which we prove below, implies that $g : \mathbb{R} \to X$ is strongly measurable.

The following lemma will be used for Pettis Theorem.

Lemma A.11. *Let X be a separable Banach space and let S' be the unit ball of the dual X'. Then X' is separable for the weak topology $\sigma(X', X)$, see Brezis [2] p.62, that is there exists a sequence $\{x'_n\}$ in S' s.t. for any $x' \in S'$ there exists a subsequence $\{x'_{n_k}\}$ s.t. for any $x \in X$ we have $\lim_{k\to\infty} \langle x'_{n_k}, x\rangle_{X'X} = \langle x', x\rangle_{X'X}$.*

Proof. Let $\{x_n\}$ be dense in X. For any n consider

$$F_n : S' \to \mathbb{R}^n \text{ defined by } F_n(x') := (\langle x', x_1\rangle_{X'X}, ..., \langle x', x_n\rangle_{X'X}).$$

Since $\mathbb{R}^n$ is separable, and so is $F_n(S')$, there exists a sequence $\{x'_{n,k}\}_k$ s.t. $\{F_n(x'_{n,k})\}_k$ is dense in $F_n(S')$. Obviously $\{x'_{n,k}\}_{n,k}$ can be put into a sequence. For any $x' \in S'$ for any n there is a k_n s.t. $|\langle x' - x'_{n,k_n}, x_i\rangle_{X'X}| < 1/n$ for all $i \leq n$. This implies that for any fixed i we have $\lim_{n\to\infty} \langle x'_{n,k_n}, x_i\rangle_{X'X} = \langle x', x_i\rangle_{X'X}$. By density this holds for any $x \in X$. □

Proposition A.12 (Pettis's Theorem)**.** *Consider $f : I \to X$. Then f is strongly measurable if and only if it is weakly measurable and almost separable valuable.*

Proof. The necessity has been already proved, so we focus on the sufficiency only. By modifying f we can assume that $f(I)$ is separable. By replacing X by a smaller space, we can assume that X is separable.
Fix now $x \in X$. Then we claim that $t \to \|f(t) - x\|$ is measurable. Indeed for any $a > 0$

$$\{t \in I : \|f(t) - x\| \leq a\} = \cap_{x' \in S'} \{t \in I : |\langle x', f(t) - x\rangle_{X'X}| \leq a\}.\}$$

Using the fact that S' is separable in the weak topology $\sigma(X', X)$ and the notation in Lemma A.11, we have

$$\{t \in I : \|f(t) - x\| \leq a\} = \cap_{n\in\mathbb{N}} \{t \in I : |\langle x'_n, f(t) - x\rangle_{X'X}| \leq a\}.$$

Since the set in the r.h.s. is measurable, we conclude that $t \to \|f(t) - x\|$ is measurable and so our claim is correct.
Consider now $n \geq 1$. Since $f(I)$ is separable there is a sequence of balls $\{B(x_j, \frac{1}{n})\}_{j\geq 0}$ whose union contains $f(I)$. Set now

$$\begin{cases} \omega_0^{(n)} := \{t : f(t) \in B(x_0, \frac{1}{n})\}, \\ \omega_j^{(n)} := \{t : f(t) \in B(x_j, \frac{1}{n})\} \setminus \cup_{k<j} \omega_k^{(n)} \end{cases}$$

and

$$f_n(t) := \sum_{j=0}^{\infty} x_j \chi_{\omega_j^{(n)}}(t).$$

Notice that $\cup_{j\geq 0} \omega_j^{(n)} = I$ and they are pairwise disjoint and measurable. By Example A.6 we know that $f_n : I \to X$ is strongly measurable. Furthermore, for any $t \in I$ there is a j s.t. $t \in \omega_j^{(n)}$ and this implies

$$\frac{1}{n} > \|f(t) - x_j\| = \|f(t) - f_n(t)\|.$$

In other words, $\|f(t) - f_n(t)\| \leq 1/n$ for any $t \in I$. Then $f_n(t) \to f(t)$ for any t, and so by Proposition A.4 the function $f : I \to X$ is strongly measurable. □

Example A.13. Consider the map $f : (0,1) \to L^\infty(0,1)$ defined by $t \xrightarrow{f} \chi_{(0,t)}$. This map is not almost separable valued. Indeed $t \neq s$ implies $\|f(t) - f(s)\|_\infty = 1$. If f was almost separable valued then there would exist a 0 measure subset E in $(0,1)$ and a countable set $\mathcal{N} = \{t_n\}_n$ in $(0,1)\backslash E$ such that for any $t \in (0,1)\backslash(E \cup \mathcal{N})$ there would exist a subsequence n_k with $f(t_{n_k}) \overset{k\to\infty}{\to} f(t)$ in $L^\infty(0,1)$. But this is impossible since $\|f(t) - f(t_{n_k})\|_\infty = 1$.
On the other hand $f : (0,1) \to L^2(0,1)$ defined in the same way, is strongly measurable. First of, since $L^2(0,1)$ is separable, it is almost separable valued. Next for any given any $w \in L^2(0,1)$ we have

$$\langle f(t), w\rangle_{L^2(0,1)} = \int_0^t w(x)dx$$

which is a continuous, and hence measurable, function. So f is also weakly measurable and hence it is strongly measurable by Pettis Theorem.

Recall that in Remark A.2 we mentioned another possible notion of measurability, that is that $f : I \to X$ could be defined as measurable if $f^{-1}(A)$ is a measurable set for any open subset $A \subseteq X$. We have the following fact.

Proposition A.14. *Consider $f : I \to X$. Then f is strongly measurable $\Leftrightarrow$ it almost separably valuable and $f^{-1}(A)$ is a measurable set for any open subset $A \subseteq X$.*

Proof. The "$\Leftarrow$" follows from the fact that for any $\mathfrak{a}$ open subset of $\mathbb{R}$ and for any $x' \in X$ the set $A = \{x \in X : \langle x, x'\rangle_{X,X'} \in \mathfrak{a}\}$ is open and for $g(t) := \langle f(t), x'\rangle_{X,X'}$ we have $f^{-1}(A) = g^{-1}(\mathfrak{a})$. So the latter being measurable it follows that g is measurable and hence f is weakly measurable. Hence by Pettis Theorem we conclude that f is strongly measurable.
We now assume that f is strongly measurable. We know from Lemma A.9 that f is almost separably valuable. Let U be an open subset of X. Let $(f_n)_n$ be a sequence in $C_c^0(I, X)$ with $f_n(t) \overset{n\to\infty}{\to} f(t)$ a.e. outside a 0 measure set $E \subset I$. Let $U_r = \{x \in X : \text{dist}(x, U^c) > r\}$. Then

$$f^{-1}(U)\backslash E = (\cup_{m\geq 1} \cup_{n\geq 1} \cap_{k\geq n} f_k^{-1}(U_{\frac{1}{m}}))\backslash E. \tag{A.4}$$

To check this, notice that if t belongs to the left hand side , then $f(t) \in U_{\frac{1}{m_0}}$ for some m_0 and, since $f_n(t) \overset{n\to\infty}{\to} f(t)$, for n large we have $f_k(t) \in U_{\frac{1}{m_1}}$ if $k \geq n$ for $m_1 > m_0$ preassigned. Viceversa if t belongs to the right hand side, then there exist n and m s.t. $f_k(t) \in U_{\frac{1}{m}}$ for all $k \geq n$. Then by $f_k(t) \overset{k\to\infty}{\to} f(t)$ it follows that $f(t) \in \overline{U_{\frac{1}{m}}}$ with the latter a subset of U. This proves (A.4). Since the r.h.s. is a measurable set, this completes the proof. □

Definition A.15 (Bochner integrability)**.** A strongly measurable function $f : I \to X$ is Bochner–integrable if there exists a sequence $(f_n(t))$ in $C_c(I, X)$ s.t.

$$\lim_{n\to\infty} \int_I \|f_n(t) - f(t)\|_X dt = 0. \tag{A.5}$$

Notice that $\|f_n(t) - f(t)\|_X$ is measurable.

Example A.16. Consider the situation of Example A.13 of a Hilbert space X with an orthonormal basis $\{e_t\}_{t\in\mathbb{R}}$ and the map $f : \mathbb{R} \to X$, which we saw is not strongly measurable and hence is not Bochner–integrable. Notice that f is Riemann integrable in any compact interval $[a, b]$ with $\int_a^b f(t)dt = 0$.
To see this recall that the Riemann integral is, if it exists, the limit

$$\int_a^b f(t)dt = \lim_{|\triangle|\to 0} \sum_{I_j\in\triangle} f(t_j)|I_j| \text{ with } t_j \in I_j \text{ arbitrary}$$

where $\triangle$ varies among all possible decompositions of $[a, b]$ and $|\triangle| = \max_{I\in\triangle} |I|$. We have

$$\| \sum_{I_j\in\triangle} e_{t_j}|I_j|\|^2 = \sum_{j,k}\langle e_{t_j}, e_{t_k}\rangle|I_j||I_k| \leq 2\sum_j |I_j||\triangle| = 2|\triangle|(b-a) \overset{|\triangle|\to 0}{\to} 0.$$

Proposition A.17. *Let $f : I \to X$ be Bochner–integrable. Then there exists an $x \in X$ s.t. if $(f_n(t))$ is a sequence in $C_c(I, X)$ satisfying* (A.5) *then we have*

$$\lim_{n\to\infty} x_n = x \text{ where } x_n := \int_I f_n(t)dt. \tag{A.6}$$

Proof. First of all we check that x_n is Cauchy. This follows immediately from (A.5) and from

$$\begin{aligned}\|x_n - x_m\|_X &= \| \int_I (f_n(t) - f_m(t))dt\|_X \leq \int_I \|f_n(t) - f_m(t))\|_X dt \\ &\leq \int_I \|f_n(t) - f(t))\|_X dt + \int_I \|f(t) - f_m(t))\|_X dt.\end{aligned}$$

Let us set $x = \lim x_n$. Let $(g_n(t))$ be another sequence in $C_c(I, X)$ satisfying (A.5). Then $\lim \int_I g_n = x$ by

$$\begin{aligned}\| \int_I g_n(t)dt - x\|_X &= \| \int_I (g_n(t) - f_n(t))dt + \int_I f_n(t)dt - x\|_X \\ &\leq \int_I \|g_n(t) - f_n(t)\|_X dt + \| \int_I f_n(t)dt - x\|_X dt \\ &\leq \int_I \|g_n(t) - f(t)\|_X dt + \int_I \|f_n(t) - f(t)\|_X dt + \| \int_I f_n(t)dt - x\|_X dt.\end{aligned}$$

□

Definition A.18. Let $f : I \to X$ be Bochner–integrable and let $x \in X$ be the corresponding element obtained from Proposition A.17. The we set $\int_I f(t)dt = x$.

Theorem A.19 (Bochner's Theorem)**.** *Let $f : I \to X$ be strongly measurable. Then f is Bochner–integrable if and only if $\|f\|$ is Lebesgue integrable. Furthermore, we have*

$$\| \int_I f(t)dt\| \leq \int_I \|f(t)\|dt. \tag{A.7}$$

Proof. Let f be Bochner–integrable. Then there is a sequence $(f_n(t))$ in $C_c(I, X)$ satisfying (A.5). We have $\|f\| \le \|f_n\| + \|f - f_n\|$. Since both functions in the r.h.s. are Lebesgue integrable and $\|f\|$ is measurable it follows that$\|f\|$ is Lebesgue integrable.

Conversely let $\|f\|$ be Lebesgue integrable. Then there exist a sequence $(g_n(t))$ in $C_c(I, \mathbb{R})$ and $g \in L^1(I)$ s.t. $\int_I |g_n(t) - \|f(t)\||dt \to 0$ and $|g_n(t)| \le g(t)$. In fact it is possible to choose such a sequence so that $\|g_n - g_m\|_{L^1(I)} < 2^{-n}$ for any n and any $m \ge n$ (just by extracting an appropriate subsequence from a starting g_n [3]). Then if we set

$$S_N(t) := \sum_{n=1}^{N} |g_n(t) - g_{n+1}(t)| \tag{A.8}$$

we have $\|S_N\|_{L^1(I)} \le 1$. Since $\{S_N(t)\}_{N\in\mathbb{N}}$ is increasing, the limit $S(t) := \lim_{n\to+\infty} S_n(t)$ remains defined, is finite a.e. and $\|S\|_{L^1(I)} \le 1$. Then $|g_n(t)| \le |g_1(t)| + S(t) =: g(t)$ everywhere, where $g \in L^1(I)$. Notice that $\lim_{n\to\infty} g_n(t)$ is convergent almost everywhere (it convergent in all points where $\lim_{n\to+\infty} S_n(t)$ is convergent). By dominated convergence it follows that this limit holds also in $L^1(I)$ and hence it is equal to $\|f\|$.
Let $(f_n(t))$ in $C_c(I, X)$ s.t. $f_n(t) \to f(t)$ a.e. (this sequence exists by the strong measurability of $f(t)$). Set

$$u_n(t) := \frac{|g_n(t)|}{\|f_n(t)\| + \frac{1}{n}} f_n(t).$$

Notice that $(u_n(t))$ is in $C_c(I, X)$. We have

$$\|u_n(t)\| \le \frac{|g_n(t)|\ \|f_n(t)\|}{\|f_n(t)\| + \frac{1}{n}} \le |g_n(t)| \le g(t).$$

We have (where the 2nd equality holds because because $\lim_{n\to\infty} g_n(t) = \|f(t)\|$ and $\lim_{n\to\infty} \|f_n(t)\| = \|f(t)\|$ a.e.)

$$\lim_{n\to\infty} u_n(t) = \lim_{n\to\infty} \frac{|g_n(t)|}{\|f_n(t)\| + \frac{1}{n}} f_n(t) = \lim_{n\to\infty} f_n(t) = f(t) \text{ a.e..}$$

Then we have

$$\lim_{n\to\infty} \|u_n(t) - f(t)\| = 0 \text{ a.e. with } \|u_n(t) - f(t)\| \le g(t) + \|f(t)\| \in L^1(I).$$

By dominated convergence we conclude

$$\lim_{n\to\infty} \int_I \|u_n(t) - f(t)\| dt = 0.$$

[3] Suppose we start with a given $\{g_n\}$. Then for any 2^{-n} there exists N_n s.t. $n_1, n_2 > N_n$ implies $\|g_{n_1} - g_{n_2}\|_{L^1(I)} < 2^{-n}$. Let now $\{\varphi(n)\}$ be a strictly increasing sequence in $\mathbb{N}$ s.t. $\varphi(n) > N_n$ for any n. Then $\|g_{\varphi(n)} - g_{\varphi(m)}\|_{L^1(I)} < 2^{-n}$ for any pair $m > n$. Rename $g_{\varphi(n)}$ as g_n.

This implies that f is Bochner–integrable. Finally, we have

$$\| \int_I f(t)dt\| = \lim_{n\to\infty} \| \int_I u_n(t)dt\| \leq \lim_{n\to\infty} \int_I \|u_n(t)\|dt = \int_I \|f(t)\|dt.$$

□

Corollary A.20 (Dominated Convergence)**.** *Consider a sequence $(f_n(t))$ of Bochner–integrable functions $I \to X$, $g : I \to \mathbb{R}$ Lebesgue integrable and let $f : I \to X$. Suppose that*

$$\|f_n(t)\| \leq g(t) \text{ for all } n$$
$$\lim_{n\to\infty} f_n(t) = f(t) \text{ for almost all } t.$$

Then f is Bochner–integrable with $\int_I f(t) = \lim_n \int_I f_n(t)$.

Proof. By Dominated Convergence in $L^1(I, \mathbb{R})$ we have $\int_I \|f(t)\| = \lim_n \int_I \|f_n(t)\|$. By Proposition A.4, as a pointwise limit a.e. of a sequence of strongly measurable functions, f is strongly measurable. By Bochner's Theorem f is Bochner–integrable. By the triangular inequality

$$\limsup_n \| \int_I (f(t) - f_n(t))\| \leq \lim_n \int_I \|f(t) - f_n(t)\| = 0$$

where the last inequality follows from $\|f(t) - f_n(t)\| \leq \|f(t)\| + g(t)$ and the standard Dominated Convergence. □

Definition A.21. Let $p \in [1, \infty]$. We denote by $L^p(I, X)$ the set of equivalence classes of strongly measurable functions $f : I \to X$ s.t. $\|f(t)\| \in L^p(I, \mathbb{R})$. We set $\|f\|_{L^p(I,X)} := \|\|f\|\|_{L^p(I,\mathbb{R})}$.

Proposition A.22. $(L^p(I, X), \| \|_{L^p})$ *is a Banach space.*

Proof. The proof is similar to the case $X = \mathbb{R}$, see [2].
(**Case** $p = \infty$). Let (f_n) be Cauchy sequence in $L^\infty(I, X)$. For any $k \geq 1$ there is a N_k s.t.

$$\|f_n - f_m\|_{L^\infty(I,X)} \leq \frac{1}{k} \text{ for all } n, m \geq N_k.$$

So there exists an $E_k \subset I$ with $|E_k| = 0$ s.t.

$$\|f_n(t) - f_m(t)\|_X \leq \frac{1}{k} \text{ for all } n, m \geq N_k \text{ and for all for } t \in I\backslash E_k.$$

Set $E := \cup_k E_k$. Then for any $t \in I\backslash E$ the sequence $(f_n(t))$ is convergent. So a function $f(t)$ remains defined with

$$\|f_n(t) - f(t)\|_X \leq \frac{1}{k} \text{ for all } n \geq N_k \text{ and for all for } t \in I\backslash E. \tag{A.9}$$

By Proposition A.4 the function f is strongly measurable. By (A.9) we have $f \in L^\infty(I,X)$ and

$$\|f_n - f\|_{L^\infty(I,X)} \leq \frac{1}{k} \text{ for all } n \geq N_k$$

and so $f_n \to f$ in $L^\infty(I,X)$.

(**Case** $p < \infty$). Let (f_n) be Cauchy sequence in $L^p(I,X)$ and let (f_{n_k}) be a subsequence with

$$\|f_{n_k} - f_{n_{k+1}}\|_{L^p(I,X)} \leq 2^{-k}.$$

Set now

$$g_l(t) = \sum_{k=1}^{l} \|f_{n_k}(t) - f_{n_{k+1}}(t)\|_X$$

Then

$$\|g_l\|_{L^p(I,\mathbb{R})} \leq 1.$$

By monotone convergence we have that $(g_l(t))_l$ converges a.e. to a $g \in L^p(I,\mathbb{R})$. Furthermore, for $2 \leq k < l$

$$\|f_{n_k}(t) - f_{n_l}(t)\|_X = \sum_{j=k}^{l-1} \|f_{n_j}(t) - f_{n_{j+1}}(t)\|_X \leq g(t) - g_{k-1}(t).$$

Then a.e. the sequence $(f_{n_k}(t))$ is Cauchy in X for a.e. t and so it converges for a.e. t to some $f(t)$. By Proposition A.4 the function f is strongly measurable. Furthermore,

$$\|f(t) - f_{n_k}(t)\|_X \leq g(t).$$

It follows that $f - f_{n_k} \in L^p(I,X)$, and so also $f \in L^p(I,X)$. Finally we claim $\|f - f_{n_k}\|_{L^p(I,X)} \to 0$. First of all we have $\|f(t) - f_{n_k}(t)\|_X \to 0$ for a.e. t and

$$\|f(t) - f_{n_k}(t)\|_X^p \leq g^p(x)$$

by dominated convergence we obtain that $\|f - f_{n_k}\|_X \to 0$ in $L^p(I,\mathbb{R})$. Hence $f_{n_k} \to f$ in $L^p(I,X)$. □

Proposition A.23. *$C_c^\infty(I,X)$ is a dense subspace of $L^p(I,X)$ for $p < \infty$.*

Proof. We split the proof in two parts. We first show that $C_c^0(I,X)$ is a dense subspace of $L^p(I,X)$ for $p < \infty$. For $p = 1$ this follows from the definition of integrable functions in Definition A.15. For $1 < p < \infty$ going through the proof of Bochner's Theorem A.19, the functions u_n considered in that proof can be taken to belong to $C_c^0(I,X)$ and converge to f in $L^p(I,X)$.

The second part of the proof consists in showing that $C_c^\infty(I,X)$ is a dense subspace of $C_c^0(I,X)$ inside $L^p(I,X)$ for $p < \infty$. Let $f \in C_c^0(I,X)$. We consider $\rho \in C_c^\infty(\mathbb{R},[0,1])$ s.t. $\int \rho(x)dx = 1$. Set $\rho_\epsilon(x) := \epsilon^{-1}\rho(x/\epsilon)$. Then for $\epsilon > 0$ small enough $\rho_\epsilon * f \in C_c^\infty(I,X)$. We

extend both f and $\rho_\epsilon * f$ on $\mathbb{R}$ setting them 0 in $\mathbb{R}\backslash I$. In this way $\rho_\epsilon * f \in C_c^\infty(\mathbb{R}, X)$ and $f \in C_c^0(\mathbb{R}, X)$ and it is enough to show that $\rho_\epsilon * f \overset{\epsilon\to 0^+}{\to} f$ in $L^p(\mathbb{R}, X)$..
We have

$$\rho_\epsilon * f(t) - f(t) = \int_{\mathbb{R}} (f(t-\epsilon s) - f(s))\rho(s)dy$$

so that, by Minkowski inequality and for $\Delta(s) := \|f(\cdot - s) - f(\cdot)\|_{L^p}$, we have

$$\|\rho_\epsilon * f(t) - f(t)\|_{L^p} \leq \int |\rho(s)|\Delta(\epsilon\ s)ds.$$

Now we have $\lim_{s\to 0} \Delta(s) = 0$ and $\Delta(s) \leq 2\|f\|_{L^p}$. So, by dominated convergence we get

$$\lim_{\epsilon\searrow 0} \|\rho_\epsilon * f - f\|_{L^p} = \lim_{\epsilon\searrow 0} \int |\rho(s)|\Delta(\epsilon\ s)ds = 0.$$

So

$$\lim_{\epsilon\searrow 0} \rho_\epsilon * f = f \text{ in } L^p(\mathbb{R}, X). \tag{A.10}$$

□

Definition A.24. We denote by $\mathcal{D}'(I, X)$ the space $\mathcal{L}(\mathcal{D}(I, \mathbb{R}), X)$.

Proposition A.25. *Let $p \in [1, \infty)$ and $f \in L^p(\mathbb{R}, X)$. Set*

$$T_h f(t) = h^{-1} \int_t^{t+h} f(s)ds \text{ for } t \in \mathbb{R} \text{ and } h \neq 0.$$

Then $T_h f \in L^p(\mathbb{R}, X) \cap L^\infty(\mathbb{R}, X) \cap C^0(\mathbb{R}, X)$ and $T_h f \overset{h\to 0}{\to} f$ in $L^p(\mathbb{R}, X)$ and for almost every t.

□

Corollary A.26. *Let $f \in L^1_{loc}(I, X)$ be such that $f = 0$ in $\mathcal{D}'(I, X)$. Then $f = 0$ a.e.*

Proof. First of all we have $\int_J f dt = 0$ for any $J \subset I$ compact. Indeed, let $(\varphi_n) \in \mathcal{D}(I)$ with $0 \leq \varphi_n \leq 1$ and $\varphi_n \to \chi_J$ a.e. Then

$$\int_J f dt = \lim_{n\to+\infty} \int_J \varphi_n f dt = 0$$

where we applied Dominated Convergence for the last equality.
Set now $\overline{f}(t) = f(t)$ in J and $\overline{f}(t) = 0$ outside J. Then $T_h\overline{f} = 0$ for all $h > 0$. Then $\overline{f}(t) = 0$ for a.e. t. So $f(t) = 0$ for a.e. $t \in J$. This implies $f(t) = 0$ for a.e. $t \in \mathbb{R}$. □

Corollary A.27. *Let $g \in L^1_{loc}(I, X)$, $t_0 \in I$, and $f \in C(I, X)$ given by $f(t) = \int_{t_0}^t g(s)ds$. Then:*

(1) $f' = g$ in $\mathcal{D}'(I, X)$;

(2) f is differentiable a.e. with $f' = g$ a.e.

Proof. It is not restrictive to consider the case $I = \mathbb{R}$ and $g \in L^1(\mathbb{R}, X)$. We have

$$T_h g(t) = h^{-1} \int_t^{t+h} g(s) ds = \frac{f(t+h) - f(t)}{h}.$$

By Proposition A.25 $T_h g \overset{h \to 0}{\to} g$ for almost every t. This yields (2).

For $\varphi \in \mathcal{D}(\mathbb{R})$ we have

$$\langle f', \varphi \rangle = - \int_{\mathbb{R}} f(t) \varphi'(t) dt.$$

Furthermore

$$\lim_{h \to 0} \frac{\varphi(t+h) - \varphi(t)}{h} = \varphi'(t) \text{ in } L^\infty(\mathbb{R}).$$

So

$$\langle f', \varphi \rangle = - \lim_{h \to 0} \int_{\mathbb{R}} f(t) \frac{\varphi(t+h) - \varphi(t)}{h} dt = - \lim_{h \to 0} \int_{\mathbb{R}} \varphi(t) \frac{f(t-h) - f(t)}{h} dt$$
$$= - \lim_{h \to 0} \int_{\mathbb{R}} \varphi(t) T_{-h} g(t) dt = \langle g, \varphi \rangle.$$

□

Definition A.28. Let $p \in [1, \infty]$. We denote by $W^{1,p}(I, X)$ the space formed by the $f \in L^p(I, X)$ s.t. $f' \in \mathcal{D}(I, X)$ is also $f' \in L^p(I, X)$ and we set $\|f\|_{W^{1,p}} = \|f\|_{L^p} + \|f'\|_{L^p}$.

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