APPLIED MATHEMATICS 27/01/2017

1) Five independent flips of a fair coin are made. Find the probability that:

(a) the first three flips are the same;

(b) the first three flips are the same or the last three flips are the same;

(c) there are at least two heads among the first three flips and at least two tails among the last three flips.

The sample space is

$$\Omega = \{H, T\}^5 .$$

(a)

$$\begin{aligned}
\mathbb{P}(\omega_1 = \omega_2 = \omega_3) &= \mathbb{P}(HHH \cup TTT) = \mathbb{P}(HHH) + \mathbb{P}(TTT) = \frac{1}{8} + \frac{1}{8} = \frac{1}{4} \\
&= \mathbb{P}(\omega_1 = \omega_2) \cdot \mathbb{P}(\omega_3 = \omega_2 | \omega_1 = \omega_2) = \frac{1}{2} \cdot \frac{1}{2}
\end{aligned}$$

(b)

$$\begin{aligned}
& \mathbb{P}(\text{the first three flips are the same or the last three flips are the same}) \\
= \; & \mathbb{P}(\text{the first three flips are the same}) + \mathbb{P}(\text{the last three flips are the same}) \\
& -\mathbb{P}(\text{the first three flips are the same} \cap \text{the last three flips are the same}) \\
= \; & \frac{1}{4} + \frac{1}{4} - \mathbb{P}(\text{the five flips are the same}) \\
= \; & \frac{1}{4} + \frac{1}{4} - \mathbb{P}(HHHH \cup TTTTT) = \frac{1}{4} + \frac{1}{4} - \frac{1}{16} = \frac{7}{16}
\end{aligned}$$

(c)

$$\mathbb{P}(HHHTT \cup HHTHT \cup HHTTH \cup HHTTT \cup HTHTT \cup THHTT) = \frac{6}{32} = \frac{3}{16}.$$

2) The height of adult women in the United States is normally distributed with mean 163.8 cm and standard deviation 6.1 cm. Find the probability that a randomly chosen woman is

(a) less than 160 cm tall;

(b) more than 170 cm tall;

(c) n women are independently and randomly selected. Find n in order to have that the sum of the heights of these selected women is larger than 20 m with probability larger than 99%.

(a)

$$\mathbb{P}(X < 160) = \left(\frac{X - 163.8}{6.1} < \frac{160 - 163.8}{6.1}\right) = \Phi\left(\frac{160 - 163.8}{6.1}\right) = 0.2676$$

(b)

$$\mathbb{P}(X > 170) = \mathbb{P}\left(\frac{X - 163.8}{6.1} > \frac{170 - 163.8}{6.1}\right) = 1 - \Phi\left(\frac{170 - 163.8}{6.1}\right) = 0.1539$$

(c) S has distribution $N\left(n\mu, (\sqrt{n}\sigma)^2\right)$ and then

$$\begin{aligned}
\mathbb{P}(S > a) \quad > \quad 0.99 &\iff \mathbb{P}\left(\frac{S - n\mu}{\sqrt{n}\sigma} > \frac{a - n\mu}{\sqrt{n}\sigma}\right) = 1 - \Phi\left(\frac{a - n\mu}{\sqrt{n}\sigma}\right) > 0.99 \\
&\iff \Phi\left(\frac{a - n\mu}{\sqrt{n}\sigma}\right) < 0.01 \\
&\iff \frac{a - n\mu}{\sqrt{n}\sigma} < z_{0.01} \iff a - n\mu < z_{0.01}\sqrt{n}\sigma
\end{aligned}$$

Posto

$$m = \sqrt{n}$$

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$$\begin{aligned}
\mu m^2 + z_{0.01}\sigma m - a &< 0 \\
\Delta &= (z_{0.01}\sigma)^2 + 4a\mu \\
m &> \frac{-z_{0.01}\sigma + \sqrt{(z_{0.01}\sigma)^2 + 4a\mu}}{2\mu} \\
n &= \left\lceil \left(\frac{-z_{0.01}\sigma + \sqrt{(z_{0.01}\sigma)^2 + 4a\mu}}{2\mu}\right)^2 \right\rceil \\
&= \left\lceil (3.4986)^2 \right\rceil = \lceil 12.2403 \rceil = 13
\end{aligned}$$

3) A question of medical importance is whether jogging leads to a reduction in one's pulse rate. To test this hypothesis, 8 nonjogging volunteers agreed to begin a 1-month jogging program. After the month their pulse rates were determined and compared with their earlier values. If the data are as follows:

Subject	1	2	3	4	5	6	7	8
Pulse Rate Before	74	86	98	102	78	84	79	70
Pulse Rate After	70	85	90	110	71	80	69	74

can we conclude that jogging has had an effect on the pulse rates?

We have

$$\mathbf{x}^{\mathrm{obs}} = (-4, -1, -8, 8, -7, -4, -10, 4)$$

with

$$\overline{x}^{\mathrm{obs}} = -2.75 \quad \text{and} \quad s_x^{\mathrm{obs}} = 6.1586.$$

So

$$t\left(\mathbf{x}^{\mathrm{obs}}\right) = \frac{\overline{x}^{\mathrm{obs}} - 0}{\frac{s_x^{\mathrm{obs}}}{\sqrt{n}}} = -1.2630$$

By testing

$$H_0 : \mu = 0 \quad \text{versus} \quad H_1 : \mu \neq 0$$

we have

$$p\ value = 2\left(1 - \Phi_{n-1}\left(\left|t\left(\mathbf{x}^{\mathrm{obs}}\right)\right|\right)\right) = 0.2470.$$

By testing

$$H_0 : \mu \geq 0 \quad \text{versus} \quad H_1 : \mu < 0$$

we have

$$p\ value = 1 - \Phi_{n-1}\left(-t\left(\mathbf{x}^{\mathrm{obs}}\right)\right) = 0.1235$$