(1

1) $f\left(B_1(0) \setminus \overline{B_{1/2}(\tfrac{1}{2})}\right) =$

$= \left\{ w \in \mathbb{C} \;\middle|\; |f^{-1}(w)| < 1 \,,\; |f^{-1}(w) - \tfrac{1}{2}| > \tfrac{1}{2} \right\}.$

$$f^{-1}(w) = \frac{iw}{-w+1}$$

$$\left|\frac{iw}{-w+1}\right| < 1 \,, \qquad |w| < |w-1| \,, \quad |w|^2 < |w-1|^2$$

$$|w|^2 < |w|^2 - 2\,\mathrm{Re}\,w + 1 \,, \qquad \mathrm{Re}\,w < \frac{1}{2}$$

Quindi $|f^{-1}(w)| < 1$ è equiv a $\mathrm{Re}\,w < \frac{1}{2}$

$$\left|\frac{iw}{-w+1} - \frac{1}{2}\right| > \frac{1}{2}$$

$$\left|iw + \frac{1}{2}(w-1)\right| > \frac{|w-1|}{2} \,, \qquad \left|\left(\frac{1}{2}+i\right)w - \frac{1}{2}\right|^2 > \frac{1}{4}|w-1|^2$$

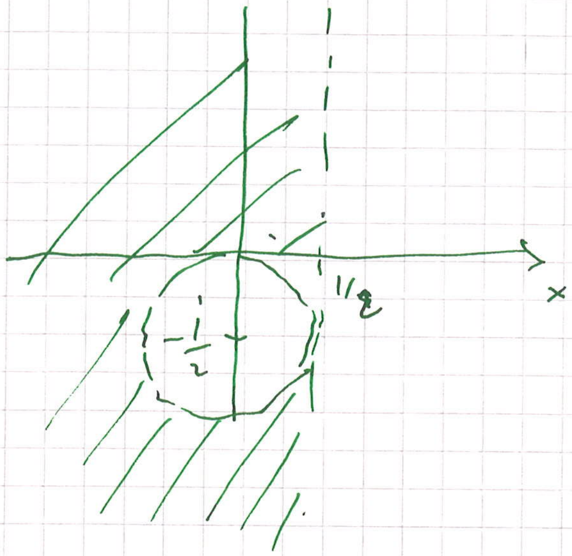
$$\left|\frac{1}{2}+i\right|^2 |w|^2 - \mathrm{Re}\left(\left(\frac{1}{2}+i\right)w\right) + \cancel{\frac{1}{4}} > \frac{1}{4}|w|^2 - \frac{1}{2}\,\mathrm{Re}\,w + \cancel{\frac{1}{4}}$$

$$\frac{5}{4}|w|^2 - \mathrm{Re}(iw) > \frac{1}{4}|w|^2 \,, \qquad |w|^2 - \mathrm{Re}(iw) > 0$$

$$\left|w + \frac{i}{2}\right|^2 > \frac{1}{4}$$

2

$$f\left(B_1(0) \setminus \overline{B_{1/2}\left(\frac{i}{2}\right)} \right) = \left\{ w \mid \operatorname{Re} w < \frac{1}{2} \right\} \setminus \overline{B_{1/2}\left(-\frac{i}{2}\right)}$$



2) $g(z) = z^9$ ha 1 zero di molt. 9 in $B_{10}(0)$

$|g| = 10^9$ su $\partial B_{10}(0)$

$$|f-g| = \left| \left(10^8 - \frac{1}{11}\right) z + 1 \right| \leq \left(10^8 - \frac{1}{11}\right) 10 + 1 =$$

$$= 10^9 + 1 - \frac{10}{11} = 10^9 - \frac{1}{11} < 10^9$$

Per Rouché, f ha 9 zeri, contati con le molt., in $B_{10}(0)$.

3)

$\dfrac{e^{-\frac{1}{z}}}{1-z}$ ha singolarità in 0 e in 1

0 è sing essenziale, 1 è un polo semplice.

$$\operatorname{Res}\left(\frac{e^{-\frac{1}{z}}}{1-z}, 1\right) = \frac{e^{-1}}{-1} = -e^{-1}.$$

Calcoliamo la serie di Laurent intorno a 0.

$$e^{-\frac{1}{z}} = \sum_{k=0}^{\infty} \frac{(-1)^k z^{-k}}{k!}, \quad \frac{1}{1-z} = \sum_{j=0}^{\infty} z^j$$

$$\frac{e^{-\frac{1}{z}}}{1-z} = \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \frac{(-1)^k z^{-k+j}}{k!} =$$

$$\ell = j-k$$

$$= \sum_{\ell=-\infty}^{+\infty} z^{\ell} \sum_{k\geq 0} \frac{(-1)^k}{k!}$$

$$j = \ell + k \geq 0 \quad (k \geq -\ell)$$

a noi interessa il coeff di z^{-1}, $\ell = -1$

quindi $j = k-1 \geq 0$ significa $k \geq 1$

$$\operatorname{Res}\left(\frac{e^{-\frac{1}{z}}}{1-z}, 0\right) = \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} - 1 =$$

$$= e^{-1} - 1$$

$$\frac{1}{2\pi i} \int \frac{e^{-\frac{1}{z}}}{1-z}\, dz = -e^{-1} + e^{-1} - 1 = -1.$$