Problem 11.8. Nonuniform probability densities

a. Write a program to simulate the simultaneous rolling of two dice. In this case the events are discrete and occur with nonuniform probability. You might wish to revisit Problem 7.12 and simulate the game of craps.

b. Write a program to verify that the sequence of random numbers $\{x_i\}$ generated by (11.30) is distributed according to the exponential distribution (11.28).

c. Generate random variables according to the probability density function

$$p(x) = \begin{cases} 2(1-x), & \text{if } 0 \leq x \leq 1; \\ 0, & \text{otherwise.} \end{cases} \tag{11.36}$$

d. Verify that the variables x and y in (11.35) are distributed according to the Gaussian distribution. What is the mean value and the standard deviation of x and of y?

e. How can you use the relations (11.35) to generate a Gaussian distribution with arbitrary mean and standard deviation?

11.6 *Neutron Transport

We now consider the application of a nonuniform probability distribution to the simulation of the transmission of neutrons through bulk matter, one of the original applications of a Monte Carlo method. Suppose that a neutron is incident on a plate of thickness t. We assume that the plate is infinite in the x and y directions and that the z axis is normal to the plate. At any point within the plate, the neutron can either be captured with probability p_c or scattered with probability p_s. These probabilities are proportional to the capture cross section and scattering cross section, respectively. If the neutron is scattered, we need to find its new direction as specified by the polar angle θ (see Figure 11.5). Because we are not interested in how far the neutron moves in the x or y direction, the value of the azimuthal angle ϕ is irrelevant.

If the neutrons are scattered equally in all directions, then the probability $p(\theta, \phi)\, d\theta d\phi$ equals $d\Omega/4\pi$, where $d\Omega$ is an infinitesimal solid angle and 4π is the total solid angle. Because $d\Omega = \sin\theta\, d\theta d\phi$, we have

$$p(\theta, \phi) = \frac{\sin\theta}{4\pi}. \tag{11.37}$$

We can find the probability density for θ and ϕ separately by integrating over the other angle. For example,

$$p(\theta) = \int_0^{2\pi} p(\theta, \phi)\, d\phi = \frac{1}{2}\sin\theta, \tag{11.38}$$

and

$$p(\phi) = \int_0^{\pi} p(\theta, \phi)\, d\theta = \frac{1}{2\pi}. \tag{11.39}$$

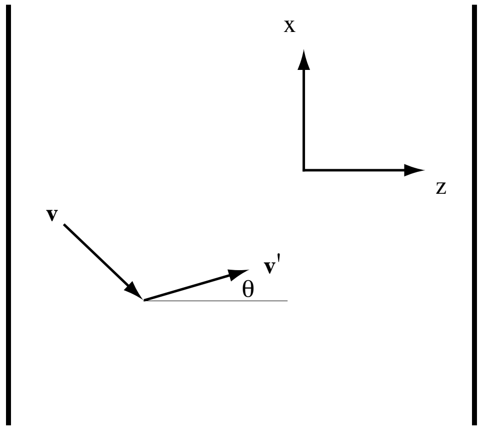

Figure 11.5: The definition of the scattering angle θ. The velocity before scattering is $\mathbf{v}$ and the velocity after scattering is $\mathbf{v}'$. The scattering angle θ is independent of $\mathbf{v}$ and is defined relative to the z axis.

Because the point probability $p(\theta, \phi)$ is the product of the probabilities $p(\theta)$ and $p(\phi)$, θ and ϕ are independent variables. Although we do not need to generate a random angle ϕ, we note that since $p(\phi)$ is a constant, ϕ can be found from the relation

$$\phi = 2\pi r. \tag{11.40}$$

To find θ according to the distribution (11.38), we substitute (11.38) in (11.24) and obtain

$$r = \frac{1}{2}\int_0^\theta \sin x\, dx \tag{11.41}$$

If we do the integration in (11.41), we find

$$\cos\theta = 1 - 2r. \tag{11.42}$$

Note that (11.40) implies that ϕ is uniformly distributed between 0 and 2π and (11.42) implies that $\cos\theta$ is uniformly distributed between -1 and $+1$.

We could invert the cosine in (11.42) to solve for θ. However, to find the z component of the path of the neutron through the plate, we need to multiply $\cos\theta$ by the path length ℓ, and hence we need $\cos\theta$ rather than θ. The path length, which is the distance traveled between subsequent scattering events, is obtained from the exponential probability density, $p(\ell) \propto e^{-\ell/\lambda}$ (see (11.28)). From (11.30), we have

$$\ell = -\lambda \ln r, \tag{11.43}$$

where λ is the mean free path.

Now we have all the necessary ingredients for calculating the probabilities for a neutron to pass through the plate, be reflected off the plate, or be captured and absorbed in the plate. The input parameters are the thickness of the plate t, the capture and scattering probabilities p_c and p_s, and the mean free path λ. We begin with $z = 0$, and implement the following steps:

1. Determine if the neutron is captured or scattered. If it is captured, then add one to the number of captured neutrons, and go to step 5.
2. If the neutron is scattered, compute $\cos\theta$ from (11.42) and ℓ from (11.43). Change the z coordinate of the neutron by $\ell\cos\theta$.
3. If $z < 0$, add one to the number of reflected neutrons. If $z > t$, add one to the number of transmitted neutrons. In either case, skip to step 5 below.
4. Repeat steps 1–3 until the fate of the neutron has been determined.
5. Repeat steps 1–4 with additional incident neutrons until sufficient data has been obtained.

Problem 11.9. Elastic neutron scattering

a. Write a program to implement the above algorithm for neutron scattering through a plate. Assume $t = 1$ and $p_c = p_s/2$. Find the transmission, reflection, and absorption probabilities for the mean free path λ equal to 0.01, 0.05, 0.1, 1, and 10. Begin with 100 incident neutrons, and increase this number until satisfactory statistics are obtained. Give a qualitative explanation of your results.

b. Choose $t = 1$, $p_c = p_s$, and $\lambda = 0.05$, and compare your results with the analogous case considered in part (a).

c. Repeat part (b) with $t = 2$ and $\lambda = 0.1$. Do the various probabilities depend on λ and t separately or only on their ratio? Answer this question before doing the simulation.

d. Draw some typical paths of the neutrons. From the nature of these paths, explain the results in parts (a)–(c). For example, how does the number of scattering events change as the absorption probability changes?

Problem 11.10. Inelastic neutron scattering

a. In Problem 11.9 we assumed elastic scattering, that is, no energy is lost during scattering. Here we assume that some of the neutron energy E is lost and that the mean free path is proportional to the speed and hence to $\sqrt{E}$. Modify your program so that a neutron loses a fraction f of its energy at each scattering event, and assume that $\lambda = \sqrt{E}$. Consider $f = 0.05, 0.1$, and 0.5, and compare your results with those found in Problem 11.9a.

b. Make a histogram for the path lengths between scattering events and plot the path length distribution function for $f = 0.1, 0.5$, and 0 (elastic scattering).