

Exercise Lecture VI:
again on classical numerical integration
(gaussian quadrature);
gaussian distribution and central limit theorem

1. 1D “classical” integration: Gaussian Quadrature

Remind the Gauss-Legendre rule for *quadrature* with non equispaced points:

$$\int_a^b f(x)dx \approx \sum_{i=1}^n w(i)f(x(i))$$

For integration in $[-1,1]$, x_i are the roots of the Legendre polynomials: abscissas and weights up to the fourth order, and the degree of the polynomial exactly integrable are listed in the following tables:

n	i	x_i	w_i	degree
1	1	0	2	1
2	1	-0.577350269189626	1	3
	2	0.577350269189626	1	
3	1	-0.774596669241483	0.555555555555556	5
	2	0	0.888888888888889	
	3	0.774596669241483	0.555555555555556	
4	1	-0.861136311594053	0.347854845137454	7
	2	-0.339981043584856	0.652145154862546	
	3	0.339981043584856	0.652145154862546	
	4	0.861136311594053	0.347854845137454	

We can transform the special points and weights for integration in an arbitrary interval $[a, b]$ with the substitution (“new” refers to $[a, b]$, “old” to $[-1, 1]$):

$$x_{new} = \frac{b-a}{2}x_{old} + \frac{b+a}{2} \quad \text{and} \quad w_{new} = \frac{b-a}{2}w_{old}$$

- (a) Consider once again the definite integral

$$I = \int_0^1 e^x dx = e - 1$$

whose numerical estimate F_N has been already calculated using (1) the trapezoidal and (2) the Simpson's rule in the previous exercise session. Now we use (3) the Gauss-Legendre quadrature. Here is listed a simple program implementing explicitly the second-order formula (`gauleg-IIorder.f90`). Verify that already at this order, the Gauss-Legendre quadrature gives a very good approximation.

- (b) A more general implementation of Gauss-Legendre is proposed in `gauleg-other.f90` which makes use of the subroutine `gauleg` from "Numerical Recipes" (but the code is self-contained, it can be used without any external routine/module/interface). Estimate the relative error

$$\epsilon = \left| \frac{\text{numeric} - \text{exact}}{\text{exact}} \right|$$

for the 3 different methods, considering e.g. $N=2, 4, 8, 16, 32, 64$. Make a log-log plot of $|\epsilon|$ as a function of N . What about the dependence of the error on N ? Can you identify the range of N where the roundoff errors are dominant? (consider the possibility of increasing the precision).

- (c) The program `gauleg_nr_test.f90` is another example of the use of the subroutine `gauleg` from "Numerical Recipes"; where the subroutine and other auxiliary routines/module/interface are external and must be compiled and linked. They are extracted from the "Numerical Recipes" library, properly simplified (the original versions contain more and more subroutines) and are listed at the end of these notes.

In order to use the routines of Numerical Recipes, you have to compile and link the main program with:

- the subroutine `gauleg` which gives points and abscissas
- `nrtype.f90` containing type declarations; - `nrutil.f90` containing modules and utilities; - `nr.f90` containing (through a module with `interfaces`) the conventions to call the subroutines with the main program

You must compile these files with the option `-c`: this produces `.mod` and `.o` (the objects). In a second step compile the main program. Finally you link all the files `.o` and produce the executable:

```
g95 -c nrtype.f90 nrutil.f90 nr.f90 gauleg.f90
g95 -c gauleg_nr_test.f90
g95 -o a.out gauleg_nr_test.o nrtype.o nrutil.o nr.o gauleg.o
```

**2. Random numbers with gaussian distribution:
the central limit theorem**

Use the central limit theorem in order to produce random numbers with gaussian distribution. Remind that given a sequence of independent random numbers r_i , their average

$$x_N = \frac{1}{N} \sum_{i=1}^N r_i$$

is distributed according to

$$P_N(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x_N - \mu)^2}{2\sigma^2}\right)$$

with $\mu = \langle r \rangle$, $\sigma^2 = (\langle r^2 \rangle - \langle r \rangle^2)/N$.
Furthermore, defining

$$z_N = \frac{x_N - \mu}{\sigma}$$

we have $\langle z_N^4 \rangle \approx 3 < z_N^2 \rangle^2$.

- (a) Use random numbers r_i uniformly distributed in $[-1,1]$, choosing $N \approx 500$ and generating at least ≈ 100 points x_N . Verify (doing an histogram) that x_N have a Gaussian distribution.
- (b) Calculate numerically $\langle x_N \rangle$ and σ_x^2 and compare them with μ and σ^2 analytically calculated.
- (c) Consider z_N and verify numerically the relationship $\langle z_N^4 \rangle \approx 3 < z_N^2 \rangle^2$.
- (d) Do again the exercise using random numbers r_i with exponential distribution ($p(r) = e^{-r}$ if $r \geq 0$, 0 elsewhere). Calculate $\langle x_N \rangle$, $\sigma_{x_N}^2$ and μ , σ^2 numerically and analytically, respectively.
- (e) Consider now random numbers r_i with distribution

$$p(r) = \frac{a}{\pi} \frac{1}{r^2 + a^2}$$

(Lorentz's), with $a = 1$, for instance. Do $\langle x \rangle$, $\langle x^2 \rangle$ and σ_x^2 exist??
Try to determine the characteristic of the distribution of the variable
sum $x_N = \sum_{i=1}^N r_i$.


```

!cccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccc
! gauleg-other.f90      P145 Numerical Recipes in Fortran
! adapted from www.cs.umbbc.edu/~squire/download/gauleg.f90
! (everything self-contained!)
! compute x(i) and w(i)  i=1,n Legendre ordinates and weights in (-1,1)
!cccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccc

```

```

subroutine gaulegf(x1, x2, x, w, n)
  implicit none
  integer, intent(in) :: n
  double precision, intent(in) :: x1, x2
  double precision, dimension(n), intent(out) :: x, w
  integer :: i, j, m
  double precision :: p1, p2, p3, pp, xl, xm, z, z1
  double precision, parameter :: eps=3.d-14

```

```

  m = (n+1)/2
  xm = 0.5d0*(x2+x1)
  xl = 0.5d0*(x2-x1)
  do i=1,m
    z = cos(3.141592654d0*(i-0.25d0)/(n+0.5d0))
    z1 = 0.0
    do while(abs(z-z1) > eps)
      p1 = 1.0d0
      p2 = 0.0d0
      do j=1,n
        p3 = p2
        p2 = p1
        p1 = ((2.0d0*j-1.0d0)*z*p2-(j-1.0d0)*p3)/j
      end do
      pp = n*(z*p1-p2)/(z*z-1.0d0)
      z1 = z
      z = z1 - p1/pp
    end do
    x(i) = xm - xl*z
    x(n+1-i) = xm + xl*z
    w(i) = (2.0d0*xl)/((1.0d0-z*z)*pp*pp)
    w(n+1-i) = w(i)
  end do

```

```

end subroutine gaulegf

```

```

program gauleg
  implicit none
  integer :: i, j
  double precision, dimension(100) :: x, w
  double precision :: sum, a, b

```

```

integer, parameter :: debug=0

print *, 'test gauleg.f90 on interval -1.0 to 1.0 ordinates, weights'
do i=1,15
  call gaulegf(-1.0d0, 1.0d0, x, w, i)
  sum = 0.0d0
  do j=1,i
    print *, 'x(',j,')=', x(j), ' w(',j,')=', w(j)
    sum = sum + w(j)
  end do
  print *, '          integrate(1.0, from -1.0 to 1.0)= ', sum
print *, ' '
end do

a = 0.5d0
b = 1.0d0
print *, 'test gauleg on integral(sin(x), from ',a,' to ',b,')'
do i=2,10
  call gaulegf(a, b, x, w, i)
  sum = 0.0d0
  do j=1,i
    if(debug>0)print *, 'x(',j,')=', x(j), ' w(',j,')=', w(j)
    sum = sum +w(j)*sin(x(j))
  end do
  print *, i, ' integral (0.5,1.0) sin(x) dx = ', sum
end do
print *, '-cos(1.0)+cos(0.5) =', -cos(b)+cos(a)
print *, 'exact should be: 0.3372802560'
print *, ' '

a = 0.5d0
b = 5.0d0
print *, 'test gauleg on integral(exp(x), from ',a,' to ',b,')'
do i=2,10
  call gaulegf(a, b, x, w, i)
  sum = 0.0d0
  do j=1,i
    if(debug>0) print *, 'x(',j,')=', x(j), ' w(',j,')=', w(j)
    sum = sum + w(j)*exp(x(j))
  end do
  print *, i, ' integral (0.5,5.0) exp(x) dx = ', sum
end do
print *, 'exp(5.0)-exp(0.5) =', exp(b)-exp(a)
print *, 'exact should be: 146.7644378'
print *, ' '
end program gauleg

```


gauleg.f90 from Numerical Recipes

```
SUBROUTINE gauleg(x1,x2,x,w)
  USE nrtype; USE nrutil, ONLY : arth,assert_eq,nrerror
  IMPLICIT NONE
  REAL(SP), INTENT(IN) :: x1,x2
  REAL(SP), DIMENSION(:), INTENT(OUT) :: x,w
  REAL(DP), PARAMETER :: EPS=3.0e-14_dp
  INTEGER(I4B) :: its,j,m,n
  INTEGER(I4B), PARAMETER :: MAXIT=10
  REAL(DP) :: x1,xm
  REAL(DP), DIMENSION((size(x)+1)/2) :: p1,p2,p3,pp,z,z1
  LOGICAL(LGT), DIMENSION((size(x)+1)/2) :: unfinished
  n=assert_eq(size(x),size(w),'gauleg')
  m=(n+1)/2
  xm=0.5_dp*(x2+x1)
  x1=0.5_dp*(x2-x1)
  z=cos(PI_D*(arth(1,1,m)-0.25_dp)/(n+0.5_dp))
  unfinished=.true.
  do its=1,MAXIT
    where (unfinished)
      p1=1.0
      p2=0.0
    end where
    do j=1,n
      where (unfinished)
        p3=p2
        p2=p1
        p1=((2.0_dp*j-1.0_dp)*z*p2-(j-1.0_dp)*p3)/j
      end where
    end do
    where (unfinished)
      pp=n*(z*p1-p2)/(z*z-1.0_dp)
      z1=z
      z=z1-p1/pp
      unfinished=(abs(z-z1) > EPS)
    end where
    if (.not. any(unfinished)) exit
  end do
  if (its == MAXIT+1) call nrerror('too many iterations in gauleg')
  x(1:m)=xm-x1*z
  x(n:n-m+1:-1)=xm+x1*z
  w(1:m)=2.0_dp*x1/((1.0_dp-z**2)*pp**2)
  w(n:n-m+1:-1)=w(1:m)
END SUBROUTINE gauleg
```


nrtype.f90 from Numerical Recipes

```
MODULE nrtype
  INTEGER, PARAMETER :: I4B = SELECTED_INT_KIND(9)
  INTEGER, PARAMETER :: I2B = SELECTED_INT_KIND(4)
  INTEGER, PARAMETER :: I1B = SELECTED_INT_KIND(2)
  INTEGER, PARAMETER :: SP = KIND(1.0)
  INTEGER, PARAMETER :: DP = KIND(1.0D0)
  INTEGER, PARAMETER :: SPC = KIND((1.0,1.0))
  INTEGER, PARAMETER :: DPC = KIND((1.0D0,1.0D0))
  INTEGER, PARAMETER :: LGT = KIND(.true.)
  REAL(SP), PARAMETER :: PI=3.141592653589793238462643383279502884197_sp
  REAL(SP), PARAMETER :: PI02=1.57079632679489661923132169163975144209858_sp
  REAL(SP), PARAMETER :: TWOPI=6.283185307179586476925286766559005768394_sp
  REAL(SP), PARAMETER :: SQRT2=1.41421356237309504880168872420969807856967_sp
  REAL(SP), PARAMETER :: EULER=0.5772156649015328606065120900824024310422_sp
  REAL(DP), PARAMETER :: PI_D=3.141592653589793238462643383279502884197_dp
  REAL(DP), PARAMETER :: PI02_D=1.57079632679489661923132169163975144209858_dp
  REAL(DP), PARAMETER :: TWOPI_D=6.283185307179586476925286766559005768394_dp
  TYPE sprs2_sp
    INTEGER(I4B) :: n,len
    REAL(SP), DIMENSION(:), POINTER :: val
    INTEGER(I4B), DIMENSION(:), POINTER :: irow
    INTEGER(I4B), DIMENSION(:), POINTER :: jcol
  END TYPE sprs2_sp
  TYPE sprs2_dp
    INTEGER(I4B) :: n,len
    REAL(DP), DIMENSION(:), POINTER :: val
    INTEGER(I4B), DIMENSION(:), POINTER :: irow
    INTEGER(I4B), DIMENSION(:), POINTER :: jcol
  END TYPE sprs2_dp
END MODULE nrtype
```

nr.f90 from Numerical Recipes

```
MODULE nr
  INTERFACE
    SUBROUTINE gauleg(x1,x2,x,w)
      USE nrtype
      REAL(SP), INTENT(IN) :: x1,x2
      REAL(SP), DIMENSION(:), INTENT(OUT) :: x,w
    END SUBROUTINE gauleg
  END INTERFACE
  ! ... the original file contains several other INTERFACES ...
END MODULE nr
```

nrutil.f90 (Here only for: array_copy, arth, assert_eq, nrerror)

```
MODULE nrutil
  USE nrtype
  IMPLICIT NONE
  INTEGER(I4B), PARAMETER :: NPAR_ARTH=16,NPAR2_ARTH=8
  INTEGER(I4B), PARAMETER :: NPAR_GEOP=4,NPAR2_GEOP=2
  INTEGER(I4B), PARAMETER :: NPAR_CUMSUM=16
  INTEGER(I4B), PARAMETER :: NPAR_CUMPROD=8
  INTEGER(I4B), PARAMETER :: NPAR_POLY=8
  INTEGER(I4B), PARAMETER :: NPAR_POLYTERM=8
  INTERFACE array_copy
    MODULE PROCEDURE array_copy_r, array_copy_d, array_copy_i
  END INTERFACE
  INTERFACE assert_eq
    MODULE PROCEDURE assert_eq2,assert_eq3,assert_eq4,assert_eqn
  END INTERFACE
  INTERFACE arth
    MODULE PROCEDURE arth_r, arth_d, arth_i
  END INTERFACE
  ! ... l'originale contiene ancora molte altre INTERFACES...
CONTAINS

  SUBROUTINE array_copy_r(src,dest,n_copied,n_not_copied)
    REAL(SP), DIMENSION(:), INTENT(IN) :: src
    REAL(SP), DIMENSION(:), INTENT(OUT) :: dest
    INTEGER(I4B), INTENT(OUT) :: n_copied, n_not_copied
    n_copied=min(size(src),size(dest))
    n_not_copied=size(src)-n_copied
    dest(1:n_copied)=src(1:n_copied)
  END SUBROUTINE array_copy_r

  SUBROUTINE array_copy_d(src,dest,n_copied,n_not_copied)
    REAL(DP), DIMENSION(:), INTENT(IN) :: src
    REAL(DP), DIMENSION(:), INTENT(OUT) :: dest
    INTEGER(I4B), INTENT(OUT) :: n_copied, n_not_copied
    n_copied=min(size(src),size(dest))
    n_not_copied=size(src)-n_copied
    dest(1:n_copied)=src(1:n_copied)
  END SUBROUTINE array_copy_d

  SUBROUTINE array_copy_i(src,dest,n_copied,n_not_copied)
    INTEGER(I4B), DIMENSION(:), INTENT(IN) :: src
    INTEGER(I4B), DIMENSION(:), INTENT(OUT) :: dest
    INTEGER(I4B), INTENT(OUT) :: n_copied, n_not_copied
    n_copied=min(size(src),size(dest))
```

```

        n_not_copied=size(src)-n_copied
        dest(1:n_copied)=src(1:n_copied)
    END SUBROUTINE array_copy_i

FUNCTION assert_eq2(n1,n2,string)
    CHARACTER(LEN=*), INTENT(IN) :: string
    INTEGER, INTENT(IN) :: n1,n2
    INTEGER :: assert_eq2
    if (n1 == n2) then
        assert_eq2=n1
    else
        write (*,*) 'nrerror: an assert_eq failed with this tag:',string
        STOP 'program terminated by assert_eq2'
    end if
END FUNCTION assert_eq2

FUNCTION assert_eq3(n1,n2,n3,string)
    CHARACTER(LEN=*), INTENT(IN) :: string
    INTEGER, INTENT(IN) :: n1,n2,n3
    INTEGER :: assert_eq3
    if (n1 == n2 .and. n2 == n3) then
        assert_eq3=n1
    else
        write (*,*) 'nrerror: an assert_eq failed with this tag:',string
        STOP 'program terminated by assert_eq3'
    end if
END FUNCTION assert_eq3

FUNCTION assert_eq4(n1,n2,n3,n4,string)
    CHARACTER(LEN=*), INTENT(IN) :: string
    INTEGER, INTENT(IN) :: n1,n2,n3,n4
    INTEGER :: assert_eq4
    if (n1 == n2 .and. n2 == n3 .and. n3 == n4) then
        assert_eq4=n1
    else
        write (*,*) 'nrerror: an assert_eq failed with this tag:',string
        STOP 'program terminated by assert_eq4'
    end if
END FUNCTION assert_eq4

FUNCTION assert_eqn(nn,string)
    CHARACTER(LEN=*), INTENT(IN) :: string
    INTEGER, DIMENSION(:), INTENT(IN) :: nn
    INTEGER :: assert_eqn
    if (all(nn(2:) == nn(1))) then
        assert_eqn=nn(1)
    end if
END FUNCTION assert_eqn

```

```

else
  write (*,*) 'nerror: an assert_eq failed with this tag:',string
  STOP 'program terminated by assert_eqn'
end if
END FUNCTION assert_eqn

```

```

SUBROUTINE nrerror(string)
  CHARACTER(LEN=*), INTENT(IN) :: string
  write (*,*) 'nerror: ',string
  STOP 'program terminated by nrerror'
END SUBROUTINE nrerror

```

```

FUNCTION arth_r(first,increment,n)
  REAL(SP), INTENT(IN) :: first,increment
  INTEGER(I4B), INTENT(IN) :: n
  REAL(SP), DIMENSION(n) :: arth_r
  INTEGER(I4B) :: k,k2
  REAL(SP) :: temp
  if (n > 0) arth_r(1)=first
  if (n <= NPAR_ARTH) then
    do k=2,n
      arth_r(k)=arth_r(k-1)+increment
    end do
  else
    do k=2,NPAR2_ARTH
      arth_r(k)=arth_r(k-1)+increment
    end do
    temp=increment*NPAR2_ARTH
    k=NPAR2_ARTH
    do
      if (k >= n) exit
      k2=k+k
      arth_r(k+1:min(k2,n))=temp+arth_r(1:min(k,n-k))
      temp=temp+temp
      k=k2
    end do
  end if
END FUNCTION arth_r

```

```

FUNCTION arth_d(first,increment,n)
  REAL(DP), INTENT(IN) :: first,increment
  INTEGER(I4B), INTENT(IN) :: n
  REAL(DP), DIMENSION(n) :: arth_d
  INTEGER(I4B) :: k,k2
  REAL(DP) :: temp
  if (n > 0) arth_d(1)=first

```

```

if (n <= NPAR_ARTH) then
  do k=2,n
    arth_d(k)=arth_d(k-1)+increment
  end do
else
  do k=2,NPAR2_ARTH
    arth_d(k)=arth_d(k-1)+increment
  end do
  temp=increment*NPAR2_ARTH
  k=NPAR2_ARTH
  do
    if (k >= n) exit
    k2=k+k
    arth_d(k+1:min(k2,n))=temp+arth_d(1:min(k,n-k))
    temp=temp+temp
    k=k2
  end do
end if
END FUNCTION arth_d

FUNCTION arth_i(first,increment,n)
  INTEGER(I4B), INTENT(IN) :: first,increment,n
  INTEGER(I4B), DIMENSION(n) :: arth_i
  INTEGER(I4B) :: k,k2,temp
  if (n > 0) arth_i(1)=first
  if (n <= NPAR_ARTH) then
    do k=2,n
      arth_i(k)=arth_i(k-1)+increment
    end do
  else
    do k=2,NPAR2_ARTH
      arth_i(k)=arth_i(k-1)+increment
    end do
    temp=increment*NPAR2_ARTH
    k=NPAR2_ARTH
    do
      if (k >= n) exit
      k2=k+k
      arth_i(k+1:min(k2,n))=temp+arth_i(1:min(k,n-k))
      temp=temp+temp
      k=k2
    end do
  end if
END FUNCTION arth_i
! .... and many other FUNCTIONS and SUBROUTINES ....
END MODULE nrutil

```