Cyber-Physical Systems

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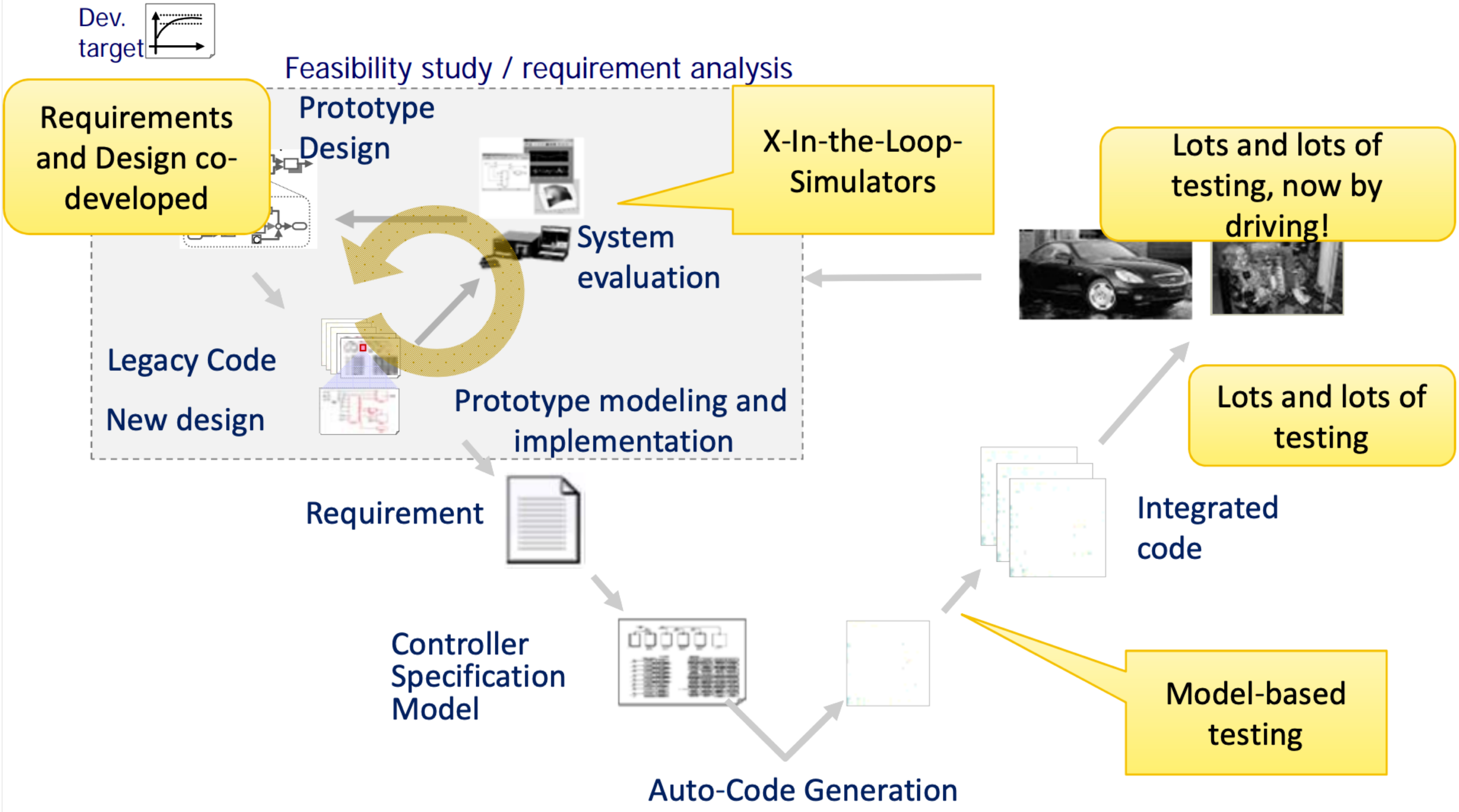
Lecture 2: Continuous Modeling


Dev.
target
Feasibility study / requirement analysis
Requirements and Design co-developed
Prototype Design
X-In-the-Loop-Simulators
Lots and lots of testing, now by driving!
System evaluation
Legacy Code
New design
Prototype modeling and implementation
Lots and lots of testing
Requirement
Integrated code
Controller Specification Model
Model-based testing
Auto-Code Generation

Models: abstractions of system dynamics

Examples of type of modeling that for CPS components:

- Modeling physical phenomena – differential equations
- Feedback control systems – time-domain modeling
- Modeling modal behavior – FSMs, hybrid automata, …
- Modeling sensors and actuators – models that help with calibration, noise elimination,
- Modeling hardware and software – capture concurrency, timing, power, …
- Modeling networks – latencies, error rates, packet loss,

Dynamic Systems

- Most natural model for describing most physical systems
- Continuous/discrete systems that continuously evolve over time
- It is represents by equation that involve the rates of change of quantities
- Quantities describe the state of the phenomena, modeled as state variables
    - Pressure, Temperature, Velocity, Acceleration, Current, Voltage, etc.
- Could include algebraic relations between state variables

Differential Equation

The state of the system is characterized by state variables, which describe the system. The rate of change is (usually) expressed with respect to time

Simple example

After drinking a cup of coffee, the amount C of caffeine in person's body follows the differential equation:

$$\frac{dC}{dt} = -\alpha C$$

$$C(t) = C_0 e^{-at}$$

Order Differential Equation

All derivatives are with respect to single independent variable, often representing time.

Order of ODE is determined by highest-order derivative of state variable function appearing in ODE

ODE with higher-order derivatives can be transformed into equivalent first-order system.

$$y^{(k)}(t) = f(t, y, y', \ldots, y^{(k-1)})$$

$$u_1(t) = y(t),\ u_2(t) = y'(t),\ \ldots,\ u_k(t) = y^{(k-1)}(t)$$

$$\begin{bmatrix} u_1'(t) \\ u_2'(t) \\ \vdots \\ u_{k-1}'(t) \\ u_k'(t) \end{bmatrix} = \begin{bmatrix} u_2(t) \\ u_3(t) \\ \vdots \\ u_k(t) \\ f(t, u_1, u_2, \ldots, u_k) \end{bmatrix}$$

For k-th order ODE

$$y^{(k)}(t) = f(t, y, y', \ldots, y^{(k-1)})$$

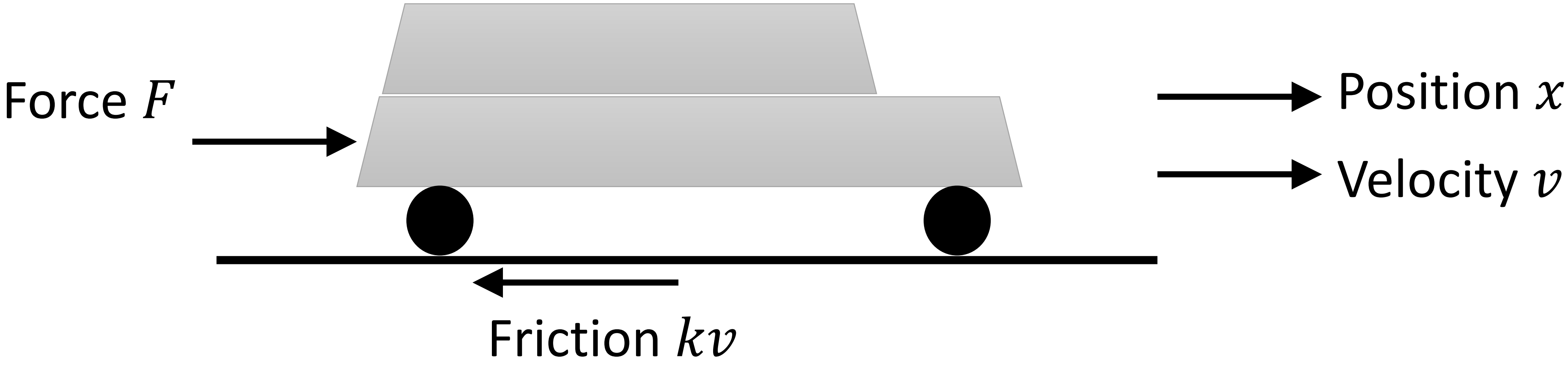
define k new unknown functions

$$u_1(t) = y(t),\ u_2(t) = y'(t),\ \ldots,\ u_k(t) = y^{(k-1)}(t)$$

Then original ODE is equivalent to first-order system

$$\begin{bmatrix} u_1'(t) \\ u_2'(t) \\ \vdots \\ u_{k-1}'(t) \\ u_k'(t) \end{bmatrix} = \begin{bmatrix} u_2(t) \\ u_3(t) \\ \vdots \\ u_k(t) \\ f(t, u_1, u_2, \ldots, u_k) \end{bmatrix}$$

Order Differential Equation



Newton's law of motion: $F = m\frac{d^2x}{dt^2} + kv\ ;\ v = \frac{dx}{dt}$

Executions of Car

- Let $\mathbb{T}$ represent a set representing time instants, i.e. $\mathbb{T} \subseteq \mathbb{R}^{\geq 0}$

- Input Signal: Function F from $\mathbb{T} \rightarrow \mathbb{R}$
  - Input signal is assumed to be continuous or piecewise-continuous

- Given an initial state (x_0, v_0) and an input signal $F(t)$, the execution of the system is defined by ***state-trajectories*** $x(t)$ and $v(t)$ (from $\mathbb{T}$ to $\mathbb{R}$) that satisfy the ***initial-value problem***:
  - $x(0) = x_0; v(0) = v_0$
  - $\dot{x} = v(t); \dot{v} = \frac{F(t) - kv(t)}{m}$

Sample Execution of Car

Suppose $\forall t\colon F(t) = 0, x_0 = 5\text{ m}, v_0 = 20\text{ m/s}, m = 1000\text{kg}, k = 50Ns/m$

- Then, we need to solve:
  - $x(0) = 5; v(0) = 20$
  - $\dot{x} = v;\ \dot{v} = -\frac{kv}{m}$

- Solution to above differential equation (solve for v first, then x):
- $v(t) = v_0 e^{-\frac{kt}{m}}; x(t) = \frac{mv_0}{k}\left(1 - e^{-\frac{kt}{m}}\right)$
- Note, as $t \to \infty$, $v(t) \to 0$, and $x(t) \to \frac{mv_0}{k}$

Sample Execution of Car with constant force

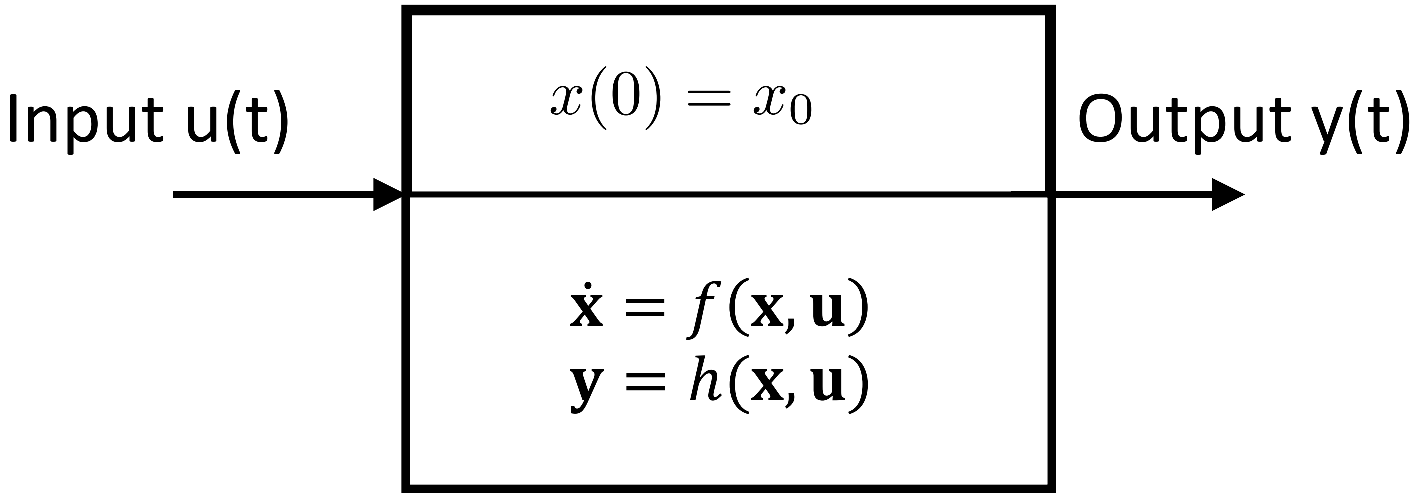
- Compute solution using Simulink/Matlab/Breach

Continuous-Time Component Definition

- Set I of real-valued input variables
- Set O or real-valued output variables
- Set X of real-valued (continuous) state variables
- Initialization $Init$ specifying a set $X_0$of initial values for states
- Dynamics: for each state variable, x, a real valued expression f over I and X
- Output Function: for each output variable, y, a real valued expression h over I and X.

Execution Definition

- Convention: $\mathbf{x} = (x_1, x_2, \dots x_n), \mathbf{y} = (y_1, y_2, \dots, y_m)$
- Given an input signal $u: \mathbb{T} \to \mathbb{R}$, an execution consists of a *differentiable* state signal $\mathbf{x}(\mathrm{t})$, and an output signal $\mathbf{y}(t)$, such that:
  1. $\mathbf{x}(0) \in X_0$
  2. For each output variable y and time t, $y(t) = h\big(u(t), x(t)\big)$
  3. For each state variable x, $\frac{d}{dt}x(t) = f(u(t), x(t))$



Existence

- There exists at least one solution $\mathbf{x}(t)$ if the function f is continuous

- Definition of continuity uses notion of distance between points
  - Euclidean distance: $d(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\|_2 = \sqrt{(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2}$

- f is uniformly continuous if for all $\epsilon > 0$, there exists a $\delta > 0$, such that for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, if $\|\mathbf{x} - \mathbf{y}\|_2 < \delta$, then $\|f(\mathbf{x}) - f(\mathbf{y})\|_2 < \epsilon$.

- Example when solution does not *globally* exist:
  - $\frac{dx}{dt} = \begin{cases} 1 \text{ if x} = 0 \\ 0 \text{ otherwise} \end{cases}$

Uniqueness

- Solution to initial value problem is unique if f is Lipschitz continuous
- Lipschitz-continuity is a stronger version of continuity: upper bounds how fast a function can change
- Function f is **Lipschitz-continuous** if there exists a constant L (called the Lipschitz constant) such that:
$$\forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^n: \|f(\mathbf{x}) - f(\mathbf{y})\| \leq L\|\mathbf{x} - \mathbf{y}\|$$
- Examples:
  - Linear functions (e.g. $x_1 - 3x_2$) are Lipschitz continuous
  - Functions: x^2, $\sqrt{x}$ are not Lipschitz continuous over $\mathbb{R}^n$
- Can restrict $\mathbb{T}$ and X to some bounded and closed set such that f is piecewise-continuous and Lipschitz to get unique solutions over such compact domains
- Rely on numerical integration schemes/solvers to obtain solutions
  - ode45, ode23, ode15, etc.

Time Invariant System

The system is time invariant because the output does not depend on the particular time the input is applied.

f does not depend on time

$$\frac{dx}{dt} = \dot{x} = f(x, u)$$

The underlying physical laws themselves do not typically depend on time.

Linear Systems

- Equation of simple car dynamics can be written compactly as:

$$\begin{bmatrix} \dot{x} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -k/m \end{bmatrix} \begin{bmatrix} x \\ v \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} [F]$$

- Letting $A = \begin{bmatrix} 0 & 1 \\ 0 & -k/m \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, we can re-write above equation in the form:

- $\dot{\mathbf{x}} = A\mathbf{x} + \mathrm{B}\mathbf{u}$, where $\mathbf{x} = [x \quad v]$, and $\mathbf{u} = [F]$

Linear Components

- Linear components model linear systems
  - f is of the form $a_1x_1 + a_2x_2 + \cdots + a_nx_n$ or compactly, $f = A\mathbf{x}$
  - h is of the form $b_1u_1 + b_2u_2 + \cdots + b_mu_m$ or compactly, $h = B\mathbf{u}$

- Linear systems have many nice properties:
  - Many analysis methods in the frequency domain (using Fourier/Laplace transform methods)
  - Superposition principle (net response to two or more stimuli is the sum of responses to each stimulus)

Solutions to Linear Systems

- ***Autonomous*** linear system has no inputs: $\dot{\mathbf{x}} = A\mathbf{x}$

- Solution of autonomous linear system can be fully characterized:
  - $\mathbf{x}(t) = e^{At}\mathbf{x}_0$
  - Computing e^{A} is easy if A is a diagonal matrix (non-zero elements are only on the diagonal)

- For a linear system with ***exogenous*** inputs?
  - $x(t) = e^{At}x_0 + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau$

- In practice, numerical integration methods outperform matrix exponential

State-Space representation

$$\dot{\mathbf{x}} = f(\mathbf{x}, \mathbf{u})$$
$$\mathbf{y} = h(\mathbf{x}, \mathbf{u})$$

Example:

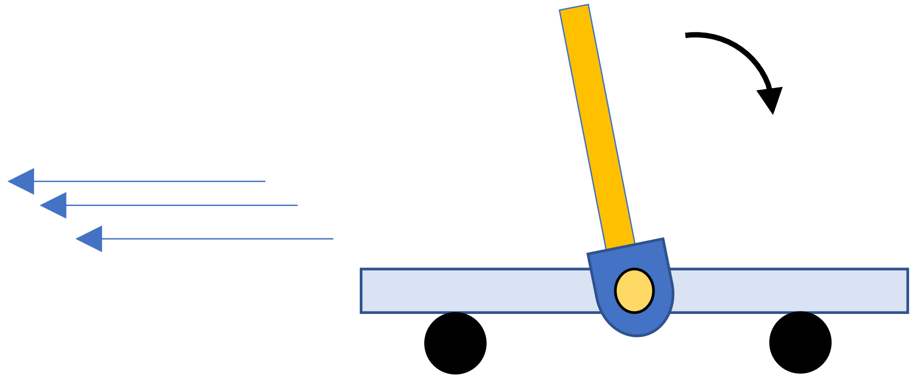
Convert

$$\dot{x} = v(t)$$
$$\dot{v} = \frac{F(t) - kv(t)}{m}$$

- It is numerically efficient to solve
- It can handle complex systems
- It allows for a more geometric understanding of dynamic systems
- It forms the basis for much of modern control theory

Stability of Systems

- Property capturing the ability of a system to return to a quiescent state after perturbation
  - Stable systems recover after disturbances, unstable systems may not
  - Almost always a desirable property for a system design
- Fundamental problem in control: design controllers to *stabilize* a system

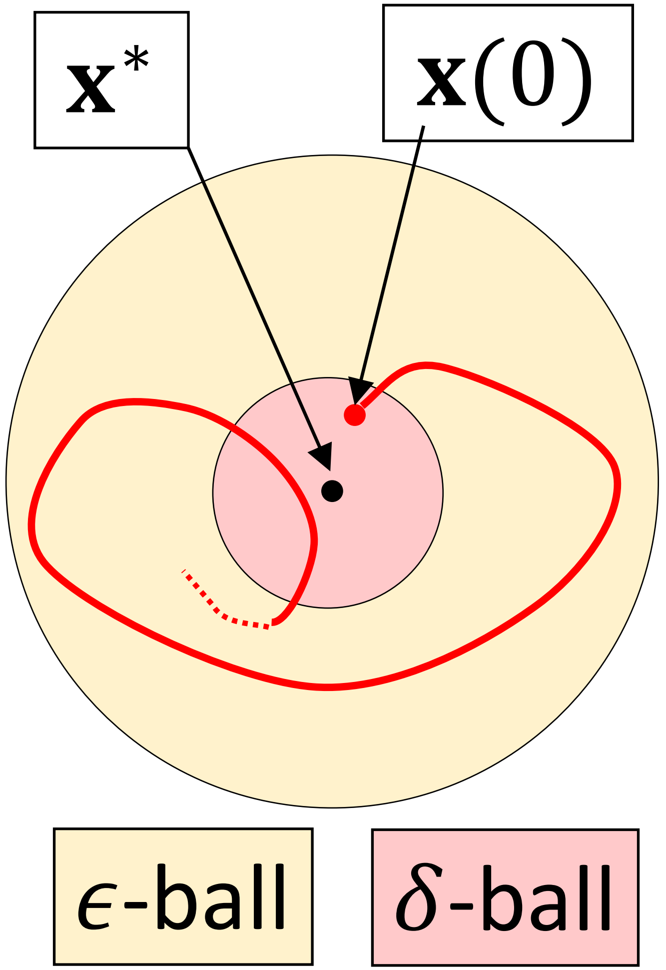


- Problem: Inverted Pendulum on a moving cart is inherently unstable, aim: keep it upright
- Solution Strategy: Move cart in direction in the same direction as the pendulum’s falling direction
- Design problem: Design a controller to stabilize the system by computing velocity and direction for cart travel

Lyapunov stability

Solutions starting δ close from equilibrium point must remain close (within ϵ) forever

- System $\dot{\mathbf{x}} = f(\mathbf{x})$ with f Lipschitz continuous
- Equilibrium point when $f(\mathbf{x})$ is zero (say $\mathbf{x}^*$)
- Equilibrium point $\mathbf{x}^*$ is Lyapunov-stable if:
  - For every $\epsilon > 0$,
    - There exists a $\delta > 0$, such that
      - if $\|\mathbf{x}(0) - \mathbf{x}^*\| < \delta$, then,
      - for every $t \geq 0$, we have $\|\mathbf{x}(t) - \mathbf{x}^*\| < \epsilon$



Asymptotic Stability

Solutions not only remain close, but also converge to the equilibrium

- System $\dot{\mathbf{x}} = f(\mathbf{x})$
- Equilibrium point $\mathbf{x}^*$ is asymptotically-stable if:
  - $\mathbf{x}^*$ is Lyapunov-stable +
  - There exists $\delta > 0$ s.t. if $||\mathbf{x}(0) - \mathbf{x}^*|| < \delta$, then $\lim_{t \to \infty} ||\mathbf{x}(t) - \mathbf{x}^*|| = 0$

Exponential Stability

Solutions not only converge to the equilibrium, but in fact converge at least as fast as a known exponential rate

- All stable linear systems are exponentially stable
- This need not be true for nonlinear systems!

- System $\dot{\mathbf{x}} = f(\mathbf{x})$
- Equilibrium point $\mathbf{x}^*$ is exponentially-stable if:
  - $\mathbf{x}^*$ is asymptotically stable +
  - There exist $\alpha > 0, \beta > 0$ s.t. if $\|\mathbf{x}(0) - \mathbf{x}^*\| < \delta$, then for all $t \geq 0$:

$$\|\mathbf{x}(t) - \mathbf{x}^*\| \leq \alpha \|\mathbf{x}(0) - \mathbf{x}^*\| e^{-\beta t}$$

Bounded-Input-Bounded-Output (BIBO) stability

If the output signal is bounded for all input signals that are bounded.

Example:

- $x(0) = x_0; v(0) = v_0$
- $\dot{x} = v(t); \dot{v} = \frac{F(t) - kv(t)}{m}$