MASTER DEGREE COURSE IN MATHEMATICS, A.Y. 2018/19
ADVANCED GEOMETRY 3 - WORKSHEET 1

To be returned by March 19th.

1. Let $F \in K[x_1, \dots, x_n]$ be a non–constant polynomial. The set $\mathbb{A}^n \setminus V(F)$ will be denoted $\mathbb{A}^n_F$. Prove that $\{\mathbb{A}^n_F | F \in K[x_1, \dots, x_n] \setminus K\}$ is a topology basis for the Zariski topology.

2. Let $n \geq 2$. Let K be an algebraically closed field. Prove that, in the n-dimensional affine space over K, $\mathbb{A}^n_K$, both any hypersurface $V(F)$ and its complementar set $\mathbb{A}^n_F$ have infinitely many points.

3. Prove that the map $\varphi : \mathbb{A}^1 \to \mathbb{A}^3$ defined by $t \to (t, t^2, t^3)$ is a homeomorphism between $\mathbb{A}^1$ and its image X, for the Zariski topology. Prove that the image X is an affine algebraic set.

4. Let K be an infinite field. Let T be a subset of the polynomial ring $K[x_0, x_1, \dots, x_n]$, and T' be the set of homogeneous components of the polynomials of T. Prove that $V_P(T) = V_P(T')$.

5. Let $\sigma : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^3$ be the Segre map, defined by

$$\sigma([a_0, a_1], [b_0, b_1]) = [a_0 b_0, a_0 b_1, a_1 b_0, a_1 b_1].$$

Prove that σ is a bijection between $\mathbb{P}^1 \times \mathbb{P}^1$ and the quadric of $\mathbb{P}^3$ of equation $x_0 x_3 - x_1 x_2 = 0$.