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A Gentle Introduction to Clifford Algebra

Laureando:
Daniele Ceravolo

Relatore:
prof. Marco Budinich

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Chapter 1

Introduction

The aim of this thesis is to show how an approach to classical and relativistic physics based on Clifford algebras can shed light on some hidden geometric meanings in our models. Although the recent revival of Clifford algebras has its origin on quantum theory, we see that the original ideas of Clifford are well grounded in classical mechanics and that they adapt well in a relativistic environment.

One does not have to go deep into the history of physics to discover the profound influence of mathematical invention. For instance, the inventions of analytic geometry and calculus was crucial to Newton's creation of classical mechanics, and tensor calculus was essential to Einstein's creation of General Theory of Relativity. For this reason, it is important to know what are the capabilities and limits of the mathematical tools we employ, and we should always know what is the geometrical meaning of the models we create.

Sadly, the profound influence of mathematics in our conception of the physical world is seldom analysed. Moreover, different areas of physics and science tend to use different symbolic systems, which have been developed through the years for solving problems in different contexts. One of the main points of this thesis is to show how we can connect different instruments by emphasizing their common geometrical significance.

Clifford algebra is not often referred as *geometric algebra* for nothing. As we are going to see, geometry has a central role in its development and consequently Clifford algebra is a valid candidate for being a universal language of physics.

This thesis is divided in three main section:

- A first chapter, where we give a definition of Clifford algebra and show some applications in two dimensional and three dimensional spaces. Furthermore, we explore the connection between complex numbers and quaternions and Clifford algebra.

- A second chapter, where we see how building a Clifford algebra of spacetime allows us to handle Lorentz transformations in a simple and elegant way. As a final example, Terrell-Penrose relativistic visualization problem is solved using Clifford Algebra.

- A third and final chapter, where we see how Clifford algebra gives us a new compact way of writing Maxwell equations, without losing any information and emphasizing their geometrical meaning.

1.1 Brief Historical Sketch

The use of complex numbers to represent vectors on a plane became well known only of the beginning of the 19th century, about three hundred years after their invention. However, the utility of complex numbers is limited: if several forces act on a body, these forces do not necessarily lie on a plane. A lot of well known mathematicians, such as Gauss and Moebius, worked unsuccessfully on the problem of generalization of complex numbers. The creation of a useful spatial analogous of complex numbers is due to William Rowan Hamilton, who, in 1843, invented the quaternion algebra. Quaternion algebra has four elements:

$$\{1, i, j, k\} \tag{1.1}$$

which satisfies the following properties:

$$i^2 = j^2 = k^2 = ijk = -1 \tag{1.2}$$

As of today, quaternions are demonstrably more efficient for handling rotations than the vectorial and matrix methods taught in standard physics courses. Although this difference hardly matters in the world of academic exercises, in more applicative fields quaternions are preferred, since they provide a faster way to compute rotations [1].

Despite the many advantages of working with quaternions, their development was blighted by some major problems. The main one was the status of vectors in quaternion algebra. Hamilton identified vectors as *pure quaternions*, i.e. quaternion with a null scalar part. At first glance, this seems fine, since pure quaternions indeed generate a three dimensional vector space: Hamilton himself invented the word *"vector"* and even nowadays i, j and k are commonly used for indicating an orthonormal set of vectors. The problem is that the subset of pure quaternions is not closed, since the product of

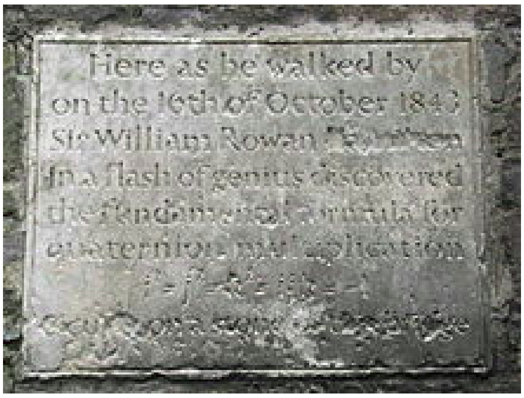

Figure 1.1: Stone Plaque in Broom Bridge in Dublin. Sir William Rowan Hamilton arrived at the quaternions algebra on 16 October 1843 while out walking with his wife, and carved the equations in stone on this bridge. His discovery of quaternions is perhaps the best-documented mathematical discovery ever.

two pure vectors does not return a pure vector. Thus, pure quaternions with the hamiltonian product cannot generate a subalgebra.

Despite their difficulties, quaternions were quite popular in their times: for instance, James Clerk Maxwell used quaternions as the basic mathematical entity.

While Hamilton was developing his quaternions another mathematician, Hermann Gunther Grassmann, who had no university education in mathematics, was developing an even more audacious algebra. Grassmann published in 1844 his *Calculus of Extension*: however, since the exposition in the book was very abstract and shrouded with mystic, its work passed almost unnoticed among his more practical minded contemporaries.

One of the core ideas introduced by Grassmann is that of outer product $\wedge$. Given three vectors a,b and c, their outer product is associative:

$$(a \wedge b) \wedge c = a \wedge (b \wedge c) \tag{1.3}$$

and anticommutative:

$$a \wedge b = -b \wedge a. \tag{1.4}$$

The outer product provides the means of encoding the plane without relying on the notion of a vector perpendicular to it. The result of the outer product is therefore neither a scalar nor a vector, but a totally new mathematical entity.

The next step was taken by William Kingdon Clifford about thirty years later. Clifford introduced a geometric product, which combined the usual scalar product and the outer product. Clifford called his algebra *geometric algebra*, acknowledging the insights of Grassmann [2].

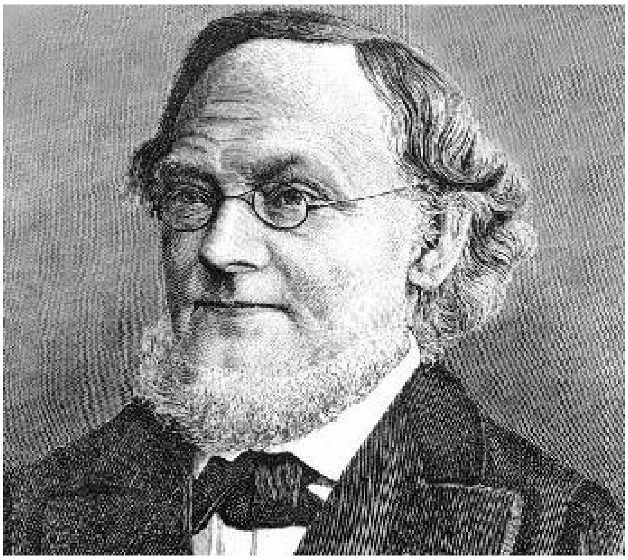

Figure 1.2: Hermann Gunther Grassmann. It should be noted that during his lifetime Grassmann was more known for his contribution to linguistic, in particular his Sanskrit expertise, than for his groundbreaking work in mathematics.

However, to quote David Hestenes: "mathematics is too important to be left to mathematicians [1]". The obscure theorems and the abstruse reasoning behind quaternions made necessary for non-mathematicians to develop their tools themselves: the formal break with quaternions and the inauguration of three-dimensional vector analysis was made independently by the physicist Josiah Gibbs and the engineer Oliver Heaviside in the early 1880s [3].

Clifford ideas remained relatively obscure for about fifty years after their conceptions, when Pauli and Dirac introduced their matrices in the quantum theory of spin. Pauli and Dirac matrices are particular representations of Clifford algebras: even if this fact was realized as soon as these matrices were introduced, all the geometrical meanings behind the algebra had already been lost by the time. This discovery gave rise to two mistaken beliefs: that the non-commutativity properties of the matrices are tied intrinsically to quantum mechanics and that matrices are crucial to understanding the properties of Clifford algebra. As we will see in the following chapters, this is not the case.

The first to draw attention to the universal nature of Clifford algebra was David Hestenes, author in 1966 of *Space-Time Algebra*. Hestenes recovered Clifford's original interpretation of the Pauli matrices and spent his career promoting applications of Clifford algebra in various areas of physics and science.

Figure 1.3: William Kingdon Clifford. Clifford took pride in his gymnastic prowess and, on one occasion, he was found "hung by his toes from the cross-bar of a weathercock on a church tower" (see [4]). Unfortunately, he died at an early age for pulmonary disease: most probably, his extreme physical exertion exacerbated his illness.

Chapter 2

Heuristic Development of Clifford Algebra

2.1 Geometric Product

Let V^n be an n-dimensional real vector space. We define a binary operation between two elements $a, b \in V^n$ called *geometric product* satisfying the following rules:

- Associativity
$$a(bc) = (ab)c; \tag{2.1}$$
where $c \in V^n$

- Left-Distributivity over Addition
$$(a + b)c = ac + bc; \tag{2.2}$$

- Right-Distributivity over Addition
$$a(b + c) = ab + ac; \tag{2.3}$$

- The square of any vector is a real scalar
$$a^2 \in \mathbb{R}. \tag{2.4}$$

The vector space V^n embedded with a geometric product defines a *Clifford algebra Cl^n*, whose properties and peculiarities we are going to explore in the next sections. Clifford algebra elements are called *multivectors*.

The key-rule of the geometric product is the last one (2.4), since it distinguishes Clifford algebra from a general associative algebra. We should also note that we do not assume our algebra to be commutative, consequently left and right distributives rules must be postulated separately.

From the quadratic form

$$(a+b)^2 = a^2 + ab + ba + b^2 \tag{2.5}$$

we build

$$ab + ba = (a+b)^2 - a^2 - b^2 \tag{2.6}$$

which is clearly symmetric and a scalar from rule (2.4). From this last expression, we define an *inner product* on V^n:

$$a \cdot b := \frac{1}{2}(ab + ba). \tag{2.7}$$

Thus, we have found a way to decompose the geometric product in a symmetric and an antisymmetric part:

$$ab = \frac{1}{2}(ab + ba) + \frac{1}{2}(ab - ba). \tag{2.8}$$

From the antisymmetric part of the geometric product we then define an *outer product*

$$a \wedge b := \frac{1}{2}(ab - ba). \tag{2.9}$$

It is curious that the symmetry of the inner product is a direct consequence of the symmetry of the addition, whereas the antisymmetry of the outer product comes from the antisymmetry of the subtraction. Since the inner product (2.7) is scalar-valued, if we force $a^2 \geq 0$, it can be identified with the standard Euclidean inner product. The fact that in our axioms we have not constrained the square a^2 to be positive will allow us to define a Clifford algebra of Minkowski spacetime without modifying any of the rules above, as we will see in later chapters.

2.2 Bivectors

We now look for a geometrical interpretation of the outer product. From the antisymmetry:

$$a \wedge b = -b \wedge a \tag{2.10}$$

we immediately see that

$$a \wedge a = 0. \tag{2.11}$$

Therefore, if

$$a \wedge b = 0 \tag{2.12}$$

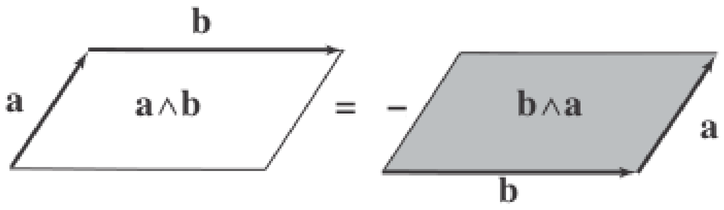

Figure 2.1: The bivectors $a \wedge b$ and $b \wedge a$. Note that the order of the arrow on the boundary determines an orientation for the parallelogram.

then a and b must be collinear. This statement is complementary to the statement that $a \cdot b = 0$ implies that a and b must be perpendicular.

We call $a \wedge b$ *bivector* or *2-vector*. Geometrically, we represent bivectors as oriented plane segments: just as vectors with the same direction can be pictured as line segments on parallel lines, so bivectors with the same direction can be pictured as plane segments on parallel planes.

The *magnitude* of $a \wedge b$ is $|a||b|\sin(\theta)$, the same as that of the area of the plane segment swept out by the two vectors. It is important to notice that the shape of the plane is not linked to any of the properties of the bivector. The *orientation* of a plane can be indicated by an arrow assigning a "sense" to the curve bounding the segment. For this reason, the two possible orientations of a bivector are usually called *clockwise* and *anticlockwise*.

2.3 Grading and Blade

By repeatedly applying the outer product we are able to build different objects. For instance, we can build an object called *3-vector* $a \wedge b \wedge c$, that can be seen as an oriented volume in three dimension. An outer product of r vector:

$$A_r = a_1 \wedge a_2 \wedge \cdots \wedge a_r \tag{2.13}$$

takes the name of *pure r-vector*. We interpret A_r as an oriented r-dimensional volume.

From the definition of outer product we get the useful (2.9) formula:

$$A_r = \frac{1}{r!} \sum (-1)^{\epsilon} a_{k_1} \dots a_{k_r} \tag{2.14}$$

where the sum runs over every permutation of the indices k_j and ϵ is $+1$ for even permutations and -1 for odd permutations. If any of the two vectors

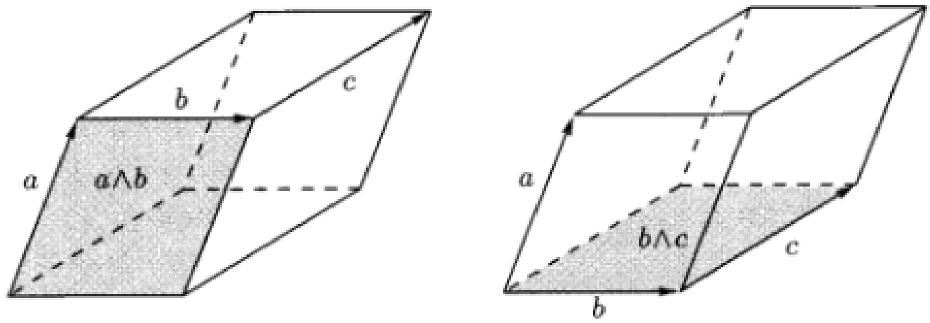

Figure 2.2: The trivector $a \wedge b \wedge c$. From this diagram, the equality $a \wedge b \wedge c = b \wedge c \wedge a$ is clear. On the left, we see the bivector $a \wedge b$ swept along c, whereas on the right $b \wedge c$ is swept along a. Both ways, we get the same trivector.

are collinear, their outer product vanishes: this is ensured by the fact that the outer product is antisymmetric, as we have seen in (2.12).

Every object that can be written as the outer product of a set of vectors is called *blade*. For example, the bivector $a \wedge b$ is a *grade 2 blade*, a trivector $a \wedge b \wedge c$ is a *grade 3 blade*, and so on.

The antisymmetrised expression (2.14) is rarely needed, since it can be proved that every blade can be written as a geometric product of orthogonal anticommuting vectors. It is straightforward to show this in a Euclidean space: given the two vectors a and b, if we take

$$b' = b - \lambda a, \tag{2.15}$$

we have that

$$a \wedge b' = a \wedge b - \lambda a \wedge a = a \wedge b. \tag{2.16}$$

Therefore the magnitude of λ is irrelevant, since we get the same bivector whatever its value. If we now form

$$a \cdot b' = a \cdot b - \lambda a \cdot a = a \cdot b - \lambda a^2 \tag{2.17}$$

and set $\lambda = \frac{a \cdot b}{a^2}$, we have $a \cdot b' = 0$. Thus we can write:

$$a \wedge b = a \wedge b' = ab'. \tag{2.18}$$

In order to get a complete set of orthogonal vectors generating the same outer product we need just to repeat the operation above.

If we consider a set of orthonormal vectors $e_1, \ldots, e_n$ in V^n, it can be proved that is possible to build from them a base for the entire algebra. For instance, a possible base of the algebra is:

$$1, \quad \{e_i\}, \quad \{e_i e_j\}_{i<j}, \quad \{e_i e_j e_k\}_{i<j<k}, \quad \ldots, \quad e_1, \ldots, e_n. \tag{2.19}$$

Every set of elements in (2.19) defines a distinct base in some grade-r subspace of Cl^n. The size of each subspace is the number of distinct combinations of r objects in a set of n, i.e. the binomial coefficients:

$$dim(Cl^n_r) = \binom{n}{r}. \tag{2.20}$$

The binomial coefficient exhibits a mirror symmetry between the grade r terms and the grade $n-r$ terms, which will return later when we will indulge on the Hodge duality (2.4.1).

From (2.20) it is simple to obtain the total dimension of the algebra:

$$dim(Cl^n) = \sum_{r=0}^{n} dim(Cl^n_r) = (1+1)^n = 2^n. \tag{2.21}$$

2.4 Multivector Algebra

Any multivector A in Cl^n can be decomposed into a sum of pure grade terms:

$$A = < A >_0 + < A >_1 + \cdots + < A >_n = \sum_r < A >_r \tag{2.22}$$

where $<>_r$ is an operator that projects onto grade r the argument. In the following chapters we will employ the abbreviation $< A >_0 = < A >$.

Given a vector a and an r-vector A_r, their geometric product will be

$$aA_r = a \cdot A_r + a \wedge A_r, \tag{2.23}$$

where we can see the generalization inner product

$$a \cdot A_r = \frac{1}{2}(aA_r - (-1)^r A_r a) = < aA_r >_{r-1}, \tag{2.24}$$

that lowers the grade of A_r by 1 unit, and the generalization of the outer product

$$a \wedge A_r = \frac{1}{2}(aA_r + (-1)^r A_r a) = < aA_r >_{r+1}, \tag{2.25}$$

that on the opposite raises the grade of A_r by 1.

The entire multivector algebra is build by repeating this process of vector multiplication.

Given two blades A_r and B_s of grade r and s respectively, their geometric product can be decomposed in the following way:

$$A_r B_s =< AB >_{r+s} + < AB >_{r+s-2} + \cdots + < AB >_{|r-s|} . \tag{2.26}$$

We retain the $\cdot$ and $\wedge$ symbols for the lowest and highest grade terms in this series:

$$A_r \cdot B_s =< AB >_{|r-s|}, \tag{2.27}$$

$$A_r \wedge B_s =< AB >_{r+s} \tag{2.28}$$

These products are associative and they respect the symmetry rules:

$$\begin{aligned} A_r \cdot B_s &= (-1)^{r(s-1)} B_s \cdot A_r \qquad \forall r \leq s, \\ A_r \wedge B_s &= (-1)^{rs} B_s \wedge A_r. \end{aligned} \tag{2.29}$$

Lastly, we define the operation of *reversion*, which invert the order of a multivector. If A is the multivector to which we apply the reversion, we call the result *revert* of A, and we denote it with $\tilde{A}$. Reversion has the following defining property:

$$(\tilde{AB}) = \tilde{B}\tilde{A}. \tag{2.30}$$

2.4.1 Pseudoscalar and Hodge Duality

The highest dimensional blade in an algebra is usually called *pseudoscalar*. The pseudoscalar unit is usually denoted with I. Through the pseudoscalar we are able to define a really useful duality operation, called *Hodge duality*. Given a grade s blade B_s in a r dimensional Clifford algebra, with $r \geq s$, we can define a dual blade by the product between the B_s and the unit pseudoscalar I:

$$B^*_{r-s} = IB_r \tag{2.31}$$

2.4.2 Basis and Reciprocal Frames

Any linearly independent set of vectors form a basis for the vectors in our algebra. Such set is often referred to as *frame*. As we have already mentioned, through the repeated use of the outer product we can build up a basis for the entire algebra. In the next chapters, we will use the symbols $e_1, ..., e_n$ for the frame vectors and we will denote the frame itself by $\{e_k\}$. We define the volume element for $\{e_k\}$ as:

$$E_n \coloneqq e_1 \wedge \cdots \wedge e_n. \tag{2.32}$$

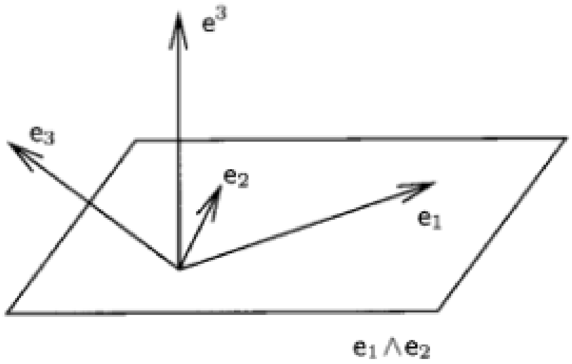

Figure 2.3: The reciprocal frame. The vectors e_1,e_2 and e_3 form a non orthonormal frame for three-dimensional space. The vector e^3 is perpendicular to $e_1 \wedge e_2$, and his magnitude is obtained by demanding $e^3 e_3 = 1$

Apparently, E_n is a multiple of the algebra pseudoscalar unit. We can associate to the frame $\{e_k\}$ another frame $\{e^k\}$ such that:

$$e_i \cdot e^j = \delta_{ij}, \quad \forall i, j = 1, \ldots, n \tag{2.33}$$

where δ_{ij} is the usual Kronecker symbol. Therefore, the reciprocal is constructed as follows:

$$e^j = (-1)^{j-1} e_1 \wedge \cdots \wedge e_{j-1} \wedge e_{j+1} \wedge \cdots \wedge e_n E_n^{-1} \tag{2.34}$$

This last expression has a simple interpretation. The vector e^j has to be perpendicular to every vector of the frame except e_j. In order to do so, we find the dual to the outer product of all the vectors in the frame $\{e_k\}$ except e_j.

2.5 Clifford Algebra of the Plane

Let us consider a two dimensional vector space, spanned by the orthonormal vectors e_1 and e_2. The full algebra Cl^2 is spanned by the basis set:

$$1, \quad \{e_1, e_2\}, \quad e_1 e_2 \tag{2.35}$$

Thus, any multivector A in Cl^2 can be written as a linear combination of the elements in (2.35):

$$A = \alpha_0 + \alpha_1 e_1 + \alpha_2 e_2 + \alpha_{12} e_1 e_2 \tag{2.36}$$

We are now going to discuss what is the role of the bivectors in the plane. We start by evaluating the following products between the base bivector and a vector of the base:

$$(e_1 \wedge e_2)e_1 = -(e_2e_1)e_1 = -e_2, \qquad (e_1 \wedge e_2)e_2 = e_1e_2e_2 = e_1, \tag{2.37}$$

$$e_1(e_1 \wedge e_2) = e_2, \qquad e_2(e_1 \wedge e_2) = -e_2(e_2e_1) = -e_1. \tag{2.38}$$

Thus, if we suppose that e_1 and e_2 forms a right-handed pair, pre-multiplication (2.37) results in an clockwise $\frac{\pi}{2}$ rotation, whereas post-multiplication (2.38) results in a anticlockwise $\frac{\pi}{2}$ rotation.

Now, if we consider the square of the base bivector:

$$(e_1 \wedge e_2)^2 = -(e_1 \wedge e_2)(e_2 \wedge e_1) = -e_1e_2e_2e_1 = -e_1e_1 = -1. \tag{2.39}$$

This means that if we post-multiply or pre-multiply twice $e_1 \wedge e_2$ to any multivector, we get the multivector turned by angle of π. Moreover, we now can give a geometrical interpretation to the imaginary unit i, which recurs often in physics.

2.5.1 Relation with Complex Numbers

The relation between a scalar and a bivector can be seen as complex number, since the unit bivector e_1e_2 acts on the left as the imaginary unit i:

$$Z = u + ve_1e_2 = u + Iv. \tag{2.40}$$

In (2.40) we have adopted $I \coloneqq e_1e_2$ to design the unit pseudoscalar. The main reason we have chosen I over the more common i is that the latter has the problem of suggesting that it commutes with other elements- which is in general not true for the pseudoscalar.

A complex number is represented by a vector in the Argand plane. We write a vector x in the subspace of Cl^2 isomorphic to $\mathbb{R}^2$ as a linear combination of e_1 and e_2:

$$x = ue_1 + ve_2 \tag{2.41}$$

If we pre-multiply x by the vectors base e_1 we obtain:

$$e_1x = ue_1e_1 + ve_1e_2 = u + Iv = Z. \tag{2.42}$$

We have thus found a simple way to connect $\mathbb{R}^2$ and $\mathbb{C}$. If we instead post-multiply x by e_1:

$$xe_1 = ue_1e_1 + ve_2e_1 = u - ve_1e_2 = u - Iv = Z^\dagger. \tag{2.43}$$

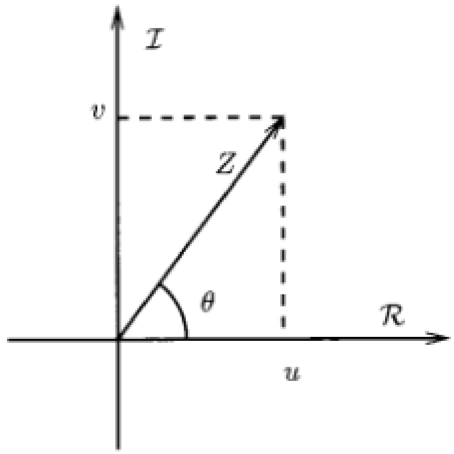

Figure 2.4: The bivectors $a \wedge b$ and $b \wedge a$. Note that the order of the arrow on the boundary determines an orientation for the parallelogram.

$Z^\dagger$ is the complex conjugate of Z. So, there is a connection between the operation of Clifford reversion and that of complex conjugation.

If we introduce another complex number W, with vector equivalent y (i.e. $W = e_1 y$), the product

$$ZW^\dagger = W^\dagger Z \tag{2.44}$$

can be expressed as:

$$W^\dagger Z = y e_1 e_1 x = yx. \tag{2.45}$$

The fact that the complex product can be seen as a geometric product should not be surprising, since historically the form of the latter was developed from the relations between complex numbers.

2.6 Clifford Algebra of Space

We consider the vector space $\mathbb{R}^3$, spanned by the orthonormal vectors e_1, e_2, e_3. Through the use of the geometric product between these vectors we are able to generate the following elements:

$$1, \quad \{e_1, e_2, e_3\}, \quad \{e_1e_2, e_2e_3, e_3e_1\}, \quad e_1e_2e_3, \tag{2.46}$$

which span the eight-dimensional Clifford algebra Cl^3. Each of the basis bivector in (2.46) shares the properties of the bivector seen in the previous section:

$$(e_1e_2)^2 = (e_2e_3)^2 = (e_3e_1)^2 = -1 \tag{2.47}$$

and each bivector generates $\frac{\pi}{2}$ rotation on its plane.

Let us now investigate the unit pseudoscalar of Cl^3, namely the trivector:

$$I = e_1 e_2 e_3. \tag{2.48}$$

First, we show that:

$$I^2 = (e_1 e_2 e_3)(e_1 e_2 e_3) = e_1 e_1 e_2 e_3 e_2 e_3 = -e_2 e_3 e_3 e_2 = -1, \tag{2.49}$$

and we see that I commutes with all the basis vectors e_k:

$$I e_k = e_1 e_2 e_3 e_k = e_k e_1 e_2 e_3 = e_k I, \qquad \forall k = 1, 2, 3 \tag{2.50}$$

therefore with every multivector of Cl^3.

Let us make a few geometrical considerations about these properties. We see that if we apply a bivector to a vector, if the latter is on the bivector plane it rotates of $\frac{\pi}{2}$, whereas if it is perpendicular to the bivector plane we get a trivector (a volume). We now have four different objects (three bivectors and a trivector) that have square -1, so we should be careful when interpreting the imaginary unit i.

2.6.1 Pauli algebra

In three dimension, we can write the full geometric product:

$$e_i e_j = e_i \cdot e_j + e_i \wedge e_j = \delta_{ij} + I \epsilon_{ijk} e_k \tag{2.51}$$

where $\epsilon_i jk$ is the Levi-Civita tensor

$$\epsilon_{jkq} = \begin{cases} 0, & \text{if among } j,\ k,\ q \text{ there are at least two equal numbers} \\ 1 & \text{if } (j,k,q) \text{ is even permutation of numbers } (1,2,3) \\ -1 & \text{if } (j,k,q) \text{ is odd permutation of numbers} \end{cases} \tag{2.52}$$

This formulation of the geometric product should be very familiar. In fact, the notorious Pauli matrices follow the same algebra:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{2.53}$$

$$\sigma_i \sigma_j = \delta_{ij} \mathbb{I}_2 + i \epsilon_{ijk} \sigma_k, \tag{2.54}$$

where $\mathbb{I}_2$ is the 2x2 identity matrix. Pauli matrices are thus a representation of the eight-dimensional Clifford algebra Cl^3.

2.6.2 Relation with Quaternions

We now explore a few interesting properties of the three independent basis bivectors in Cl^3. We introduce the following labels:

$$B_1 = e_2e_3 \quad B_2 = e_3e_1 \quad B_3 = e_1e_2, . \tag{2.55}$$

We find that their product satisfies

$$B_iB_j = -\delta_{ij} - i\epsilon_{ijk}B_k \tag{2.56}$$

From (2.56) we clearly see that there is a strong connection with the geometric product of vectors. In three dimension, the subalgebra of bivectors is clearly closed -but if we get on four or higher dimension (2.56) is no more correct. Moreover, in higher dimension we are able form grade 4 vectors from the geometric product of two bivectors.

We can see that the basis bivectors satisfy the following rules:

$$B_i^2 = -1, \qquad \forall i = 1, 2, 3 \tag{2.57}$$

$$B_iB_j = -B_jB_i \qquad \forall i \neq j. \tag{2.58}$$

These rules above are shared by the quaternion algebra generators $\{i, j, k\}$. Hamilton imposed the condition $ijk = -1$, but

$$B_1B_2B_3 = e_2e_3e_3e_1e_1e_2 = +1 \tag{2.59}$$

The difference between $\{B_1, B_2, B_3\}$ and $\{i, j, k\}$ is that the latter are a left-handed set of bivectors, whereas we have build the former in order to be right-handed. Therefore, in order to find an isomorphism between the bivectors subalgebra and the quaternions algebra we need to flip a sign on one of their generator:

$$B_1 \leftrightarrow i \qquad B_2 \leftrightarrow -j \qquad B_3 \leftrightarrow k. \tag{2.60}$$

2.7 Reflections

Clifford algebra offers a simple way to compute reflections and rotations. Let us consider a unit vector n (unit in the sense that $n^2 = 1$) in the euclidean space. Now, if we take a vector a:

$$\begin{aligned} a &= n^2a \\ &= nna \\ &= n(n \cdot a) + n(n \wedge a) \\ &= a_\parallel + a_\perp, \end{aligned} \tag{2.61}$$

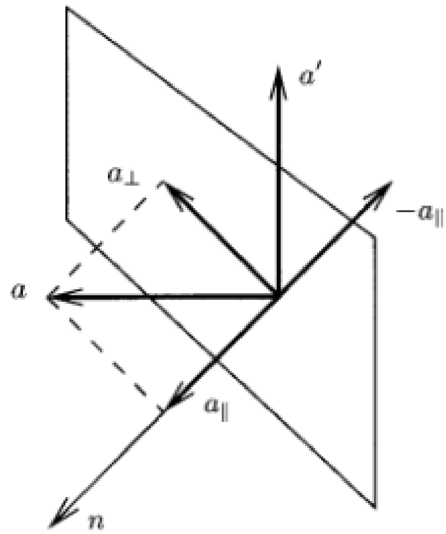

Figure 2.5: The bivectors $a \wedge b$ and $b \wedge a$. Note that the order of the arrow on the boundary determines an orientation for the parallelogram.

where

$$a_{\|} = nn \cdot a, \qquad a_{\perp} = nn \wedge a. \tag{2.62}$$

It is clear that $a_{\|}$ is the projection of a onto n, therefore the second term $a_{\perp}$ must necessarily be the perpendicular component. Anyway, we can check in a simple way that $a_{\perp}$ is perpendicular to n:

$$n \cdot a_{\perp} =< nnn \wedge a >=< n \wedge a >= 0. \tag{2.63}$$

The result of reflecting a vector a in the plane orthogonal to n is the vector $a' = -a_{\|} + a_{\perp}$. In our algebra a' can also be written as:

$$\begin{aligned} a' &= -a_{\|} + a_{\perp} \\ &= -nn \cdot a + nn \wedge a \\ &= -na \cdot n - na \wedge n \\ &= -nan. \end{aligned} \tag{2.64}$$

Thus, the geometric product offers a compact way of writing of reflections. It is noteworthy that we did not make assumption about the dimension of the vectors space, therefore (2.64) is valid in spaces of any dimension.

We can check (2.64) by showing that it leaves angles and lengths unchanged. Given two vectors a and b, if we both reflect them on the same plane, the scalar product between the resulting vectors should be equal to that between a and b:

$$(-nan) \cdot (-nbn) =< nannbn >=< nabn >=< abnn >= a \cdot b. \tag{2.65}$$

Now, suppose we have a bivector $a \wedge b$. If we reflect both a and b on a plane perpendicular to n, then

$$\begin{aligned}(-nan) \wedge (-nbn) &= \frac{1}{2}(nannbn - nbnnan) \\ &= \frac{1}{2}n(ab - ba)n \\ &= n(a \wedge b)n.\end{aligned} \tag{2.66}$$

Therefore, we see that bivectors follow the same transformation rules for reflection as vectors, *except for a change of sign.*

2.7.1 Cross Product

In the three dimensional Euclidean algebra the Hodge dual of a bivector is a vector, and vice-versa. This special property of Cl^3 enables us to define the familiar cross product of two vectors as follows:

$$a \times b = -Ia \wedge b. \tag{2.67}$$

The fact that $a \times b$ is actually the Hodge dual of a bivector is at the origin of the different reflection rules for *polar* and *axial* vectors. The former are the "standard" vectors, so they follow (2.64), whereas the latter follow (2.66), since they are generated by a duality operation from bivectors.

2.8 Rotations

The *Cartan-Dieudonné* theorem is central in our treatment of rotations in Clifford algebra: it states that *every rotation can be decomposed in a pair of reflection.* If m and n are the unit vectors orthogonal to the two hyperplanes of reflections, the rotation takes place in the $m \wedge n$ plane. Moreover, the angle of rotation is two times the angle between m and n.

Let us take a look at the action of two successive reflection on a vector a. We start by performing the first reflection

$$b = -mam, \tag{2.68}$$

and then the second

$$c = -n(-mam)n = nmamn. \tag{2.69}$$

If we define

$$R = nm, \tag{2.70}$$

Then (2.69) can be written as:

$$c = Ra\tilde{R}. \tag{2.71}$$

R is called *rotor*. In our treatment, we have not made any assumption on the dimension of the space we are working in. Therefore, we can safely say that (2.71) is valid on any dimension.

The rotor R trivially fulfils the identity

$$R\tilde{R} = \tilde{R}R = 1, \tag{2.72}$$

which fundamental in order to prove that transformation of the kind of $a \to Ra\tilde{R}$ preserve lengths and angles.

We can find another representation of the rotor. We start by noticing that:

$$R = nm = n \cdot m + n \wedge m = \cos(\theta) + n \wedge m. \tag{2.73}$$

In order to continue, we have first to compute $(n \wedge m)^2$:

$$\begin{aligned}(n \wedge m)^2 &= (nm - n \cdot m)(n \cdot m - mn) \\ &= nm\cos(\theta) - nmmn - \cos(\theta)^2 + mn\cos(\theta) \\ &= -1 - \cos(\theta)^2 + \cos(\theta)(nm + mn) \\ &= \cos(\theta)^2 - 1 \\ &= -\sin(\theta)^2\end{aligned} \tag{2.74}$$

We now define a bivector B satisfying the following properties:

$$B = \frac{m \wedge n}{\sin(\theta)}, \qquad B^2 = -1. \tag{2.75}$$

Finally, we can represent the rotor R in term of B:

$$R = \cos(\theta) - B\sin(\theta) \tag{2.76}$$

which is simply the polar decomposition of a complex number, with the imaginary unit i replaced by B. Thus, we can write:

$$R = e^{-B\theta} \tag{2.77}$$

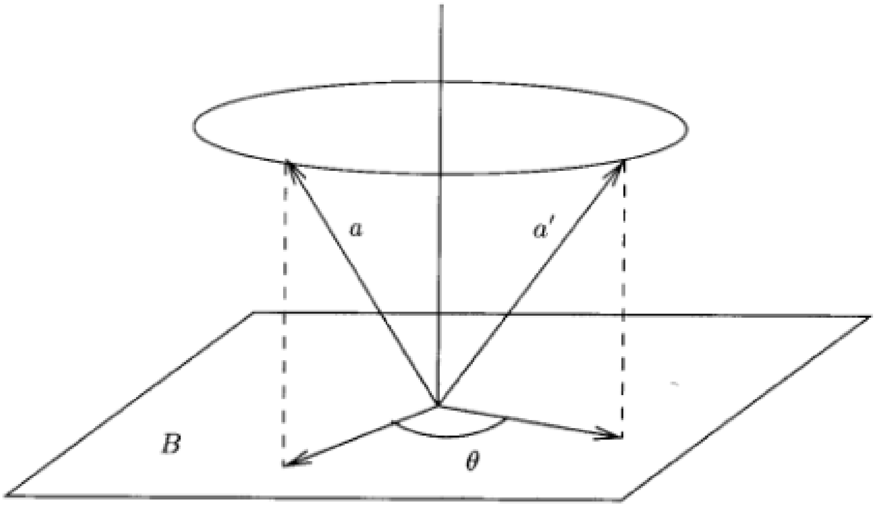

Figure 2.6: Rotation in three dimensions. The vector a is rotated in $a' = Ra\tilde{R}$, where $R = e^{-B\theta/2}$. The direction of rotation is specified by the bivector B.

with the exponential defined in terms of its power series in the usual way.

We now recall that the angle of rotation is two times that between m and n. Therefore, if we want to describe a rotation through an angle θ, R should be rewritten as

$$R = e^{-B\theta/2}. \tag{2.78}$$

At last, we have obtained a new formula for the rotation of a multivector through an angle θ in the plane specified by B:

$$a \to e^{-B\theta/2} a e^{B\theta/2}. \tag{2.79}$$

2.9 Differentiation

Lastly, we define a *vector derivative* with the symbol ∇, or, in two or three dimensions, $\boldsymbol{\nabla}$. We treat ∇ as a grade-1 object (i.e. a vector) in our Clifford algebra. The inner product between the vector derivative and a vector a is called *directional derivative* in the a direction:

$$a \cdot \nabla F(x) = \lim_{\varepsilon \to 0} \frac{F(x + \varepsilon a) - F(x)}{\varepsilon} \tag{2.80}$$

If we now define a frame $\{e_k\}$ with reciprocal frame $\{e^k\}$, the vector derivative can be written

$$\nabla = \sum_k e^k \frac{\partial}{\partial x^k} \tag{2.81}$$

or, in a compact way:

$$\nabla = e^k \partial_k. \tag{2.82}$$

From (2.81) we clearly see that the vector derivative combines both the algebraic properties of a vector and the operator properties of partial derivatives. Since the choice of the frame $\{e_k\}$ is arbitrary, the vector derivative is independent from it. For a more rigorous treatment of vector derivatives, we refer the reader to [2].

2.9.1 Multivectorial Derivative

We can extend the vector derivative to a generic multivector field A. We have:

$$\nabla A = e^k \partial_k A \tag{2.83}$$

and for a r-blade field A_r:

$$\begin{aligned} \nabla \cdot A &= < A_r >_{r-1} \\ \nabla \wedge A &= < A_r >_{r+1} \end{aligned} \tag{2.84}$$

which defines an *inner derivative* and an *outer derivative* respectively. The inner and outer derivative are a generalization of the common divergence and curl (but again, for a more complete treatment the reader should refer to [2]).

An immediate and important result is that the outer derivative of an outer derivative is always null:

$$\begin{aligned} \nabla \wedge (\nabla \wedge A) &= e^i \wedge \partial_i (e^j \partial_j A) \\ &= e^i \wedge e^j \wedge (\partial_i \partial_i A) \\ &= 0. \end{aligned} \tag{2.85}$$

In the last equation, we have used the fact that $e^i \wedge e^j$ is antisymmetric, whereas $\partial_i \partial_j$ is symmetric (partial derivatives commute). Similarly, by using the duality relationship we obtain that the inner product of an inner product is null:

$$\nabla \cdot (\nabla A) = 0. \tag{2.86}$$

2.9.2 Spacetime Derivative

Lastly, we define a spacetime derivative, that will be central in the following chapters. We consider an orthonormal frame $\{\gamma_\mu\}$, with associated coordinates x^μ. We can write:

$$\nabla = \gamma^\mu \partial_\mu = \gamma_0 \partial_0 + \gamma^i \partial_i. \tag{2.87}$$

If we post-multiply ∇ by γ_0 , we see that:

$$\nabla \gamma_0 = \partial_0 + \gamma^i \gamma_0 \partial_i = \partial_0 - \boldsymbol{\nabla}, \tag{2.88}$$

where

$$\boldsymbol{\nabla} = \gamma_0 \wedge \nabla = \boldsymbol{\sigma_i} \partial_i \tag{2.89}$$

is the vector derivative in the *relative space* derived by the choice of γ_0 (see section (3.3)).

Similarly, we obtain:

$$\gamma_0 \nabla = \partial_0 + \boldsymbol{\nabla} \tag{2.90}$$

Therefore, it is easy to see that the vector derivative satisfies the relation:

$$\begin{aligned} \nabla^2 &= \nabla \gamma_0 \gamma_0 \nabla \\ &= (\partial_0 - \boldsymbol{\nabla})(\partial_0 + \boldsymbol{\nabla}) \\ &= {\partial_0}^2 - \boldsymbol{\nabla}^2. \end{aligned} \tag{2.91}$$

Thus, the square of the vector derivative in a spacetime frame is actually the *d'Alambert* operator, which is commonly used in classical electrodynamics to describe electromagnetic waves.

Chapter 3

Spacetime Algebra

Many aspects of special relativity become clearer in the language of Clifford algebra. The Clifford algebra $Cl^{1,3}$, build from the relativistic Minkowski spacetime, is commonly referred in literature as *spacetime algebra*.

Vectors in Minkowski spacetime do not have a positive-definite norm: we can have negative squares as well as positive squares vectors. A quadrivector $x = (ct, x_1, x_2, x_3)$ is called *timelike* if $x^2 > 0$, *spacelike* if $x^2 < 0$ and *lightlike* if $x^2 = 0$. The interval d between two events:

$$d := (ct)^2 - r^2 \tag{3.1}$$

is always the same in any inertial system. The invariance of d is the starting point for any formulation of special relativity and need to be incorporated in our algebraic construction of spacetime.

We start by taking four orthogonal vectors $\{\gamma_0, \gamma_1, \gamma_2, \gamma_3\}$, satisfying the following relations:

$$\gamma_0{}^2 = 1 \quad \gamma_0 \cdot \gamma_i = 0 \quad \gamma_i \cdot \gamma_j = -\delta_{i,k} \quad where \quad i, j = 1, 2, 3 \tag{3.2}$$

That can be summarised by the standard notations

$$\gamma_\mu \gamma_\nu = \eta_{\mu\nu} \quad where \quad \eta_{\mu\nu} = diag(+ - - -) \quad and \quad \mu, \nu = 0, 1, 2, 3 \tag{3.3}$$

We have chosen the *particle physics* choice of signature, which gives spacelike vectors negative norm. The vectors of the set $\{\gamma_\mu\}$ are dimensionless, as can be seen by their square. Since we are in a space of mixed signature, we must distinguish between a frame and its reciprocal: given $\{\gamma_\mu\}$, the vectors of its reciprocal $\{\gamma^\mu\}$ are defined by $\gamma^0 = \gamma_0$ and $\gamma^i = -\gamma_i$.

3.1 Spacetime Bivectors and Pseudoscalar

There are six bivectors in our algebra, that fall in two different classes: those which have a timelike component γ_0 and those which do not. If we take two orthogonal vectors a and b, their outer product:

$$\begin{aligned}(a \wedge b)^2 &= (a \cdot b + a \wedge b)^2 \\ &= (ab)^2 \\ &= -abba \\ &= -a^2b^2\end{aligned} \tag{3.4}$$

It is easy then to see that bivectors that don't have a timelike component have negative squares:

$$(\gamma_i \wedge \gamma_j)^2 = -\gamma_i^2\gamma_j^2 = -1, \tag{3.5}$$

so they generate rotations in their plane, as we have have seen in section (2.8). However, for bivectors containing a timelike component we have:

$$(\gamma_i \wedge \gamma_j)^2 = -\gamma_i^2\gamma_0^2 = +1. \tag{3.6}$$

Bivectors of the kind of (3.6) have some new properties. One immediate result is that:

$$\begin{aligned}e^{\alpha\gamma_1\gamma_0} &= 1 + \alpha\gamma_1\gamma_0 + \frac{\alpha^2}{2!} + \frac{\alpha^3}{3!}\gamma_1\gamma_0 + \dots \\ &= \cosh(\alpha) + \sinh(\alpha)\gamma_1\gamma_0.\end{aligned} \tag{3.7}$$

This shows us that we are dealing with *hyperbolic geometry.*

3.2 Spacetime Frames

We assume we have an initial observer with constant velocity v. We associate to the velocity a timelike vector e_0 of a spacetime set $\{e_\mu\}$. The remaining vectors e_i of the frame are chosen in order to form an orthonormal right-handed set perpendicular to $e_0 = v$. Thus, the vectors in $\{e_\mu\}$ satisfy:

$$e_\nu \cdot e_\mu = \eta_{\mu\nu} \tag{3.8}$$

where $\eta_{\mu\nu} = diag(+---)$. We now assume that any event in spacetime can be described by a set of coordinates:

$$x^\mu = x \cdot e^\mu \tag{3.9}$$

3.3 Relative Vectors

An event can be decomposed in this frame in the following way:

$$x = te_0 + x^i e_i \tag{3.10}$$

Now suppose we have our event is an object at rest in our frame. We can see:

$$x^i e_i = x \cdot e^\mu e_\mu - x \cdot e^0 e_0 = x - x \cdot vv = x \wedge vv \tag{3.11}$$

It is now plain to see what does it mean to wedge x with v: we gain the projection of x in the relative frame inertial to v. We call this quantity the *relative vector* $\mathbf{x}$:

$$\mathbf{x} \coloneqq x \wedge v \tag{3.12}$$

Thus:

$$xv = x \cdot v + x \wedge v = t + \mathbf{x} \tag{3.13}$$

We can now see manifestly that the x^2 does not depend on our choice of frame:

$$\begin{aligned} x^2 &= xvvx \\ &= (x \cdot v + x \wedge v)(x \cdot v + v \wedge x) \\ &= (t + \mathbf{x})(t - \mathbf{x}) \\ &= t^2 - \mathbf{x}^2 \end{aligned} \tag{3.14}$$

If we had a second observer in a different frame the vector split would be different, but as we can see from above the interval x^2 is the same for all the observers.

From now on, we are going to indicate all. spacetime bivectors, including relative vectors, in bold fonts.

3.4 Even Subalgebra

Each observer sees a set of relative vectors, that we can model as spacetime bivectors. We want to know what are their algebraic properties. For simplicity, we choose a base set $\{\boldsymbol{\sigma}_i\}$ defined as:

$$\boldsymbol{\sigma}_i = \gamma_i \gamma_0 \tag{3.15}$$

The bivectors $\{\boldsymbol{\sigma}_i\}$ are orthogonal and represent a set of timelike planes. It is straightforward to see that their commutator satisfies the following property:

$$[\boldsymbol{\sigma_i}, \boldsymbol{\sigma_j}] = 2I\epsilon_{ijk}\boldsymbol{\sigma_k}, \tag{3.16}$$

where I is the unit pseudoscalar defined in the previous chapter. We can see that I is the pseudoscalar of both the spacetime algebra and the subalgebra generated by $\{\boldsymbol{\sigma}_i\}$:

$$\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2\boldsymbol{\sigma}_3 = \gamma_1\gamma_0\gamma_2\gamma_0\gamma_3\gamma_0 = -\gamma_1\gamma_0\gamma_2\gamma_3 = I \tag{3.17}$$

The subalgebra of even grade terms is called *even subalgebra*. It contains a scalar term, six bivector terms and a pseudoscalar term. The bivector terms are then divided in three timelike and three spacelike vectors, which in turns become relative vectors and bivectors. This is called *spacetime split* and it is observer-dependant (meaning that different velocity vectors generate different splits).

3.5 Relative Velocity

Suppose we have a particle moving with with velocity $u(t) = \dot{x}(t)$, $u^2 = 1$, in our frame. Then, if we investigate what is the spacetime split of the velocity we get:

$$uv = \frac{d}{d\tau}(xv) = \frac{d}{d\tau}(t + \mathbf{x}). \tag{3.18}$$

Therefore:

$$\frac{dt}{d\tau} = x \cdot v \quad \frac{d\mathbf{x}}{d\tau} = x \wedge v. \tag{3.19}$$

Thus we found a simple way to compute the relative velocity $\mathbf{u}$:

$$\mathbf{u} = \frac{d\mathbf{x}}{dt} = \frac{d\mathbf{x}}{d\tau}\frac{d\tau}{dt} = \frac{x \wedge v}{x \cdot v}. \tag{3.20}$$

This formulation for relative vectors shows their antisymmetric nature: we can switch u and v and the second observer will measure exactly the same relative speed, but with opposite sign. In addition, we can see that the bivector $\mathbf{u}$ is homogeneous,meaning that we can rescale u and v and always obtain the same bivector[1]. This latter fact shows that relative vectors are independent of evolution parameters: they are solely determined by the spacetime trajectories themselves.

We can see that:

$$\frac{(u \wedge v)^2}{(u \cdot v)^2} = 1 - \frac{1}{(u \cdot v)^2} < 1. \tag{3.21}$$

[1]There is a similarity between our treatment and projective geometry. For additional readings on the subject, we refer to [2]

Therefore, the magnitude of every relative vector must be inferior of the magnitude of the speed of light.

Finally, if we decompose the Lorentz Factor γ

$$\begin{aligned}\gamma^{-2} &= 1 - \mathbf{u}^2 \\ &= 1 - (u \cdot v)^{-2}[uv - u \cdot v][vu - v \cdot u] \\ &= (u \cdot v)^{-2},\end{aligned} \tag{3.22}$$

we find that

$$\gamma = u \cdot v. \tag{3.23}$$

This gives us a new way of decomposing the velocity:

$$u = uvv = (u \cdot v + u \wedge v)v = \gamma(1 - \mathbf{u}^2)v \tag{3.24}$$

Geometrically, this last decomposition means that we have split the vector in a part γv parallel to v and a part $\gamma \mathbf{u} v$ in the rest space of v

3.6 Momentum and Wave Vectors

We define the wavevector k of a photon:

$$k = \omega\gamma_0 + k^i\gamma_i, \tag{3.25}$$

where ω is the frequency of the photon and k_i are the components of the wavevector measured in the γ_0 frame. From quantum theory, we know that the energy and the momentum of a photon are:

$$E = \hbar\omega \qquad \mathbf{p} = \hbar\mathbf{k}. \tag{3.26}$$

With our definition of k the energy momentum vector is simply

$$p = \hbar k. \tag{3.27}$$

An observer with velocity v as opposed to γ_0 will measure the following energy and momentum:

$$E = p \cdot v \qquad \mathbf{p} = p \wedge v. \tag{3.28}$$

We take this definition as correct even for massive particles: if we take an object with rest mass m and velocity u will have an energy-momentum vector $p = mu$. A spacetime split with v yields:

$$pv = p \cdot v + p \wedge v = E + \mathbf{p} \tag{3.29}$$

With this definition the moment is related to velocity by:

$$\mathbf{p} = p \cdot v = mu \cdot v\mathbf{u} = \gamma m\mathbf{u}. \tag{3.30}$$

Last, we see that we have an invariant quantity:

$$m^2 = p^2 = pvvp = (E + \mathbf{p})(E - \mathbf{p}) = E^2 - \mathbf{p}^2. \tag{3.31}$$

It is straightforward to see that for a photon with wavevector k, $k^2 = 0$ this implies:

$$0 = kvvk = (\omega + \mathbf{k})(\omega - \mathbf{k}) = \omega^2 + \mathbf{k}^2 \tag{3.32}$$

Thus in every frame the relation

$$\omega = |\mathbf{k}| \tag{3.33}$$

holds.

3.7 Lorentz Transformations

A linear function that relates inertial system and leaves spacetime interval invariant is called *Lorentz transformation*. There is a striking similarity between Lorentz transformations and 3D spatial rotations seen in section 2.8: a spatial rotation is a linear function which preserves the distance between points, whereas a Lorentz transformation preserves spacetime intervals. In this section, we are going to show how both can be written with the same notation.

Lorentz transformations are usually expressed as coordinates transformations. Suppose we have two frames, S and S', whose 1 and 2 axes coincide, but S' moves at a scalar velocity βc along the 3 axis of S. If S and S' coincide at $t = 0$ we get the following transformations of the coordinates:

$$t' = \gamma(t - \beta x^3) \quad x^{1'} = x^1 \quad x^{2'} = x^2 \quad x^{3'} = \gamma(x^3 - \beta t) \tag{3.34}$$

where γ is the Lorentz factor $\gamma = \frac{1}{\sqrt{1-\beta^2}}$ and β is the velocity in unit of c. We want to convert the transformation of the coordinates to transformation of the frame vectors. In order to do this, we first have to decompose a vector x in the two frames $\{e_\mu\}$ and $\{e'_\mu\}$:

$$x = x^\mu e_\mu = x'^\mu e'_\mu. \tag{3.35}$$

If we now concentrate on the 0 and 3 components (1 and 2 are the same) we obtain:

$$te_0 + x^3 e_3 = t' e'_0 + x'^3 e_3, \tag{3.36}$$

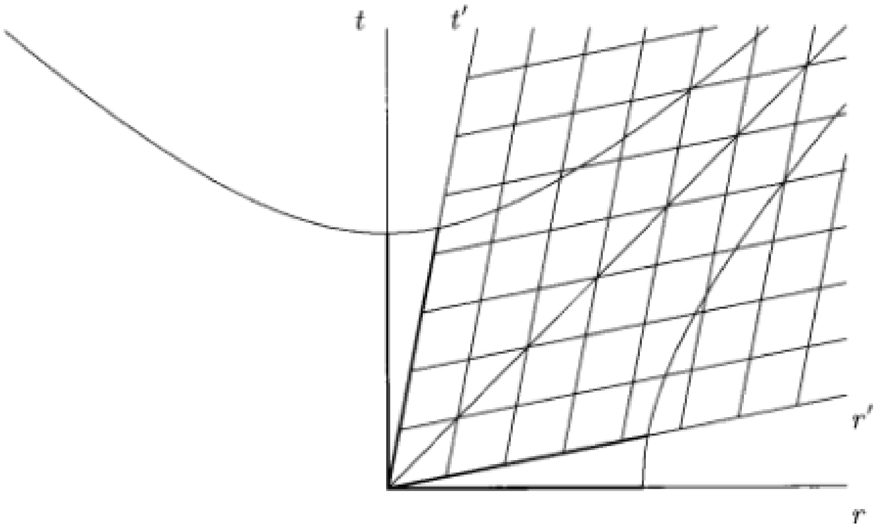

Figure 3.1: A Lorentz transformation in a Minkowski diagram. The magnitude of the vector is unchanged by the transformation: as the geometry of the spacetime plane is Lorentzian, vectors of constant magnitude do not lie on a circle but on an hyperbolae. As we can see, the transformed axes define a new coordinate grid.

and from this equation we obtain the following relations between the frame vectors:

$$e'_0 = \gamma(e_0 + \beta e_3) \quad e'_3 = \gamma(e_3 + \beta e_0) \tag{3.37}$$

Now we are going to see why from these equations it arises an *hyperbolic geometry*. We introduce an angle α, defined by:

$$\tanh(\alpha) = \beta \tag{3.38}$$

We see that:

$$\gamma = (1 - \tanh^2(\alpha))^{-1/2} = \cosh(\alpha) \tag{3.39}$$

Thus the vector e'_0 now is:

$$\begin{aligned} e'_0 &= \cosh(\alpha)e_0 + \sinh(\alpha)e_3 \\ &= (\cosh(\alpha) + \sinh(\alpha)e_3e_0)e_0 \\ &= e^{\alpha e_3 e_0}e_0 \end{aligned} \tag{3.40}$$

And, similarly:

$$e'_3 = e^{\alpha e_3 e_0}e_3 \tag{3.41}$$

Since the other two vectors in the frames are not affected by the Lorentz transformation, we can express the relationship between the two frames by:

$$e'_\mu = Re_\mu \tilde{R}, \quad R = e^{\alpha e_3 e_0} \tag{3.42}$$

The components 1 and 2 commute with R, leading to identity, while e_0 and e_3 are changed.

3.7.1 Addition of Velocities

As an example, let us now consider the addiction of velocities of two objects in special relativity. Suppose we have two points moving with constant velocities v_1 and v_2 in the $\{\gamma_\mu\}$ frame

$$v_1 = e^{\alpha_1 \gamma_1 \gamma_0 /2} \gamma_0 e^{-\alpha_1 \gamma_1 \gamma_0 /2} = e^{\alpha_1 \gamma_1 \gamma_0} \gamma_0 \tag{3.43}$$

$$v_2 = e^{\alpha_2 \gamma_1 \gamma_0 /2} \gamma_0 e^{-\alpha_2 \gamma_1 \gamma_0 /2} = e^{\alpha_2 \gamma_1 \gamma_0} \gamma_0. \tag{3.44}$$

Using (3.20) we obtain their relative velocity:

$$\frac{v_1 \wedge v_2}{v_1 \cdot v_2} = \frac{< e^{(\alpha_1+\alpha_2)\gamma_1\gamma_0} >_2}{< e^{(\alpha_1+\alpha_2)\gamma_1\gamma_0} >_0} = \frac{\sinh(\alpha_1+\alpha_2)\gamma_1\gamma_0}{\cosh(\alpha_1+\alpha_2)}, \tag{3.45}$$

therefore, both the observers measure a velocity:

$$\tanh(\alpha_1+\alpha_2) = \frac{\tanh(\alpha_1)+\tanh(\alpha_2)}{1+\tanh(\alpha_1)\tanh(\alpha_2)}. \tag{3.46}$$

Thus the addiction of collinear velocities is done by adding the hyperbolic angle. By replacing the tanh factors by the scalar velocities $u = c\tanh(\alpha)$ we get the usual formula:

$$u' = \frac{u_1+u_2}{1+\frac{u_1 u_2}{c^2}}. \tag{3.47}$$

Therefore, addition of velocities in spacetime algebra can be seen as a generalized rotation in an hyperbolic space.

3.7.2 The Lorentz Group

We now investigate the structure of the Lorentz group using Clifford algebra. The full Lorentz group consists of the transformation group for vectors which preserves lengths and angles, like reflections and rotations. As we saw in section (2.8), a reflection of an element a on the hyperplane perpendicular to n can be written as:

$$a \rightarrow nan^{-1} \tag{3.48}$$

This is true for both timelike and spacelike cases, but not for the null case, as the inverse of the null vector doesn't exist. The difference between a timelike and a spacelike reflection is that the first generates a time-reversal transformation, while the latter preserves time-ordering. A pair of spacelike or timelike transformations always preserve time-ordering, while a pair of a spacelike and a timelike transformation does not.

We can then separate the Lorentz Group in four disjoint sections:

	Parity Preserving	Space Reflection
Time Order Preserving	Type 1	Type 2
Time Reversal	Type 3	Type 4

Type 1 and type 4 transformations are derived by even numbers of reflections, and the other types by adding from a single extra reflection to a type 1 transformation.

Type 1 and type 4 can be written in the form:

$$a \rightarrow \psi a \psi^{-1}, \tag{3.49}$$

where ψ is an even multivector satisfying the following equation:

$$\psi\tilde{\psi} = \widetilde{(\psi\tilde{\psi})} \tag{3.50}$$

With the latter condition $\psi\tilde{\psi}$ may only have scalar and pseudoscalar terms:

$$\psi\tilde{\psi} = \alpha_1 + I\alpha_2 = \rho e^{I\beta}. \tag{3.51}$$

In this formula, ρ must be different from zero, otherwise ψ^{-1} cannot exist. We are finally able to build the rotor R:

$$R = \psi(\rho e^{I\beta})^{-1/2} \tag{3.52}$$

The condition $R\tilde{R} = 1$,necessary to keep the magnitude of the transformed vector invariant, is trivially fulfilled.

We now have:

$$\psi = \rho^{1/2} e^{I\beta/2} R \tag{3.53}$$

$$\psi^{-1} = \rho^{-1/2} e^{-I\beta/2} \tilde{R}, \tag{3.54}$$

therefore, we can write our transformation:

$$a \rightarrow \psi a \psi^{-1} = e^{I\beta} R a \tilde{R}. \tag{3.55}$$

In order to make the right-hand side of (3.55) a vector, since $Ra\tilde{R}$ is a vector (it's equal to its own reverse), β must be equal either to 0, which leads to a type 1 transformation, or π, which leads to a type 4.

Type 1 transformations $a \rightarrow Ra\tilde{R}$ are called *proper orthochronous*. Proper orthochronous transformations preserve casual ordering and parity.

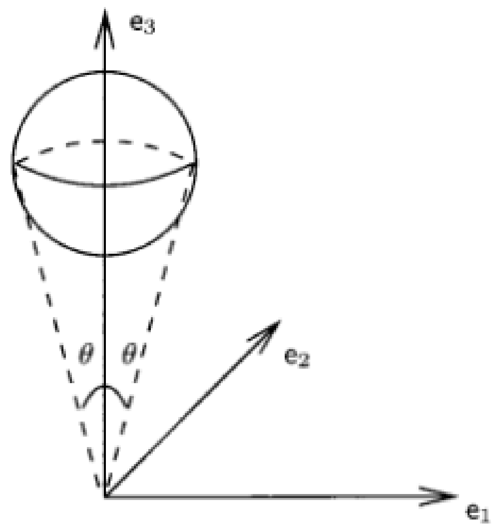

Figure 3.2: Relativistic visualization of a sphere.

3.8 Relativistic Visualization

Now we are going to see why it is important to separate the concept of observer from that of coordinate lattice. The visualization of relativistic object is a rather tricky subject: the first authors to point out that Lorentz contractions are invisible to the eye were Terrell and Penrose, more than fifty years after the birth of special relativity! [5] [6]

We want to know how two observers in relative motion see a sphere. We put the sphere and the first observer at rest in the γ_0 frame. This observer sees the edge of the sphere as a circle described by the unit vectors:

$$\mathbf{n} = \sin(\theta)(\cos(\phi)\boldsymbol{\sigma}_1 + \sin(\phi)\boldsymbol{\sigma}_2) + n\cos(\theta)\boldsymbol{\sigma}_3 \quad 0 \leq \phi < 2\pi \tag{3.56}$$

The angle θ is fixed, so the sphere encompass an angle of 2π in our vision and is centred on the 3 axis. We see that the incoming photons are described by the equation:

$$n = (1 - \mathbf{n})\gamma_0 \tag{3.57}$$

The second observer has relative velocity $\beta = \tanh(\alpha)$ in the 1 axis. Therefore:

$$v = R\gamma_0\tilde{R} \quad R = e^{\alpha\gamma_1\gamma_0} \tag{3.58}$$

and we see, using the results of the previous section, that:

$$
\begin{aligned}
n' &= Rn\tilde{R} \\
&= R(1-\mathbf{n})\gamma_0\tilde{R} \\
&= R\gamma_0\tilde{R} - R\mathbf{n}\gamma_0\tilde{R} \\
&= v - R[\sin(\theta)(\cos(\phi)\gamma_1 + \sin(\phi)\gamma_2) + \cos(\theta)\gamma_3]\tilde{R} \\
&= \cosh(\alpha)\gamma_0 + \sinh(\alpha)\gamma_1 - [\sin(\theta)\cos(\phi)R\gamma_1\tilde{R} + \sin(\theta)\sin(\phi)\gamma_2 + \cos(\theta)\gamma_3] \\
&= \cosh(\alpha)\gamma_0 + \sinh(\alpha)\gamma_1 - [\sin(\theta)\cos(\phi)(\cosh(\alpha)\gamma_1 + \sinh(\alpha)\gamma_0) \\
&\quad + \sin(\theta)\sin(\phi)\gamma_2 + \cos(\theta)\gamma_3] \\
&= \cosh(\alpha)(1+\tanh(\alpha)\sin(\theta)\cos(\phi))\gamma_0 - \cosh(\alpha)(\sin(\theta)\cos(\phi) + \tanh(\alpha))\gamma_1 \\
&\quad - \sin(\theta)\sin(\phi)\gamma_2 - \cos(\theta)\gamma_3)
\end{aligned}
\tag{3.59}
$$

We get the relative vector $\mathbf{n}'$ by the usual split:

$$\mathbf{n}' = \frac{n \wedge \gamma_0}{n \cdot \gamma_0} \tag{3.60}$$

By recalling that $\gamma_0 \wedge \gamma_0 = 0$, $\gamma_i \wedge \gamma_0 = \gamma_i\gamma_0 = \boldsymbol{\sigma}_i$ and that $\tanh(\alpha) = \beta$,(3.60) becomes:

$$\mathbf{n}' = \frac{\cosh(\alpha)(\sin(\theta)\cos(\phi) + \beta)\boldsymbol{\sigma}_1 + \sin(\theta)\sin(\phi)\boldsymbol{\sigma}_2 + \cos(\theta)\boldsymbol{\sigma}_3}{\cosh(\alpha)(1+\beta\sin(\theta)\cos(\phi))} \tag{3.61}$$

If we now take the vector

$$\mathbf{c} = \boldsymbol{\sigma}_3 + \sinh(\alpha)\cos(\theta)\boldsymbol{\sigma}_1 \tag{3.62}$$

we see that it satisfies the equation:

$$\mathbf{c} \cdot \mathbf{n} = \cosh(\alpha)\cos(\theta) \tag{3.63}$$

which is independent of ϕ. Therefore, according to the second observer all points on the edge of the sphere subtend the same angle to **c**. Thus **c** must point at the centre of the circle and the *second observer still sees the edge of the sphere as circular*. It follows that there is no observable contraction along the direction of motion. The only difference visible to the eye is that the moving observer sees the angular diameter of the sphere reduced from 2θ to $2\theta'$, where):

$$\cos(\theta') = \frac{\cos(\theta)\cosh(\alpha)}{(1+\sinh^2(\alpha)\cos^2(\theta))^{1/2}} \tag{3.64}$$

$$\sin(\theta') = \frac{tan(\theta)}{\gamma} \tag{3.65}$$

Chapter 4

Electromagnetism in Clifford Algebra

The electromagnetic field can be written as a single bivector field F in our Clifford spacetime algebra. If we represent the charged current density with J, Maxwell equations can be written:

$$\nabla F = J, \tag{4.1}$$

where ∇ is the spacetime vector derivative. By using (2.88), it is straightforward to obtain the split:

$$\nabla \cdot F = J \qquad \nabla \wedge F = 0. \tag{4.2}$$

In tensor language, these latter equations corresponds to the spacetime equations:

$$\partial_\mu F^{\mu\nu} = \eta^\nu \qquad \epsilon^{\mu\nu\rho\sigma}\partial_\nu F_{\rho\sigma} = 0. \tag{4.3}$$

It is worth mentioning that (4.3) are as compact a formulation of Maxwell equations as tensor algebra can achieve. The same is true for differential forms: only geometric algebra can encode Maxwell equations into the single equation $\nabla F = J$ [7].

F is called *Faraday bivector*. If we choose a particular timelike direction γ_0, F can be written as:

$$F = \mathbf{E} + I\mathbf{B}, \tag{4.4}$$

where $\mathbf{E}$ and $\mathbf{B}$ are the electric and magnetic field:

$$\mathbf{E} = \frac{F - \gamma_0 F \gamma_0}{2} \qquad I\mathbf{B} = \frac{F + \gamma_0 F \gamma_0}{2}. \tag{4.5}$$

This shows clearly how the split of F in $\mathbf{E}$ and $\mathbf{B}$ depends on the observer velocity γ_0.

Furthermore, we can write the charged current density J in the form:

$$J = J\gamma_0\gamma_0 = (J \cdot \gamma_0 + J \wedge \gamma_0)\gamma_0 = (\rho - \mathbf{J})\gamma_0. \tag{4.6}$$

where ρ and $\mathbf{J}$ are the charge density and the electric current density respectively:

$$\rho \equiv J \cdot \gamma_0 \qquad \mathbf{J} \equiv J \wedge \gamma_0. \tag{4.7}$$

Premultiplying (4.1) by γ_0, we have:

$$(\partial_0 + \boldsymbol{\nabla})(\mathbf{E} + I\mathbf{B}) = \rho - \mathbf{J}, \tag{4.8}$$

therefore

$$\partial_0\mathbf{E} + \boldsymbol{\nabla}\mathbf{E} + I(\partial_0\mathbf{B} + \boldsymbol{\nabla}\mathbf{B}) = \rho - \mathbf{J}. \tag{4.9}$$

In this latter equation, by separating the same grade terms in both sides of the equation, we obtain the four equations:

$$\boldsymbol{\nabla} \cdot \mathbf{E} = \rho, \tag{4.10}$$

$$\partial_0\mathbf{E} + I\boldsymbol{\nabla} \wedge \mathbf{B} = -\mathbf{J}, \tag{4.11}$$

$$I\partial_0\mathbf{B} + \boldsymbol{\nabla} \wedge \mathbf{E} = 0, \tag{4.12}$$

$$I\boldsymbol{\nabla} \cdot \mathbf{B} = 0. \tag{4.13}$$

Recalling the definition of product we gave in section (2.7.1), these latter equations can be easily converted into the familiar form of Maxwell equations.

Another immediate result can be obtained by premultiplying (4.1) by ∇:

$$\nabla^2 F = \nabla J = \nabla \cdot J + \nabla \wedge J. \tag{4.14}$$

Since ∇^2 is a scalar operator and F is a bivector, the right-hand side of (4.14) must be a bivector. It follows that the scalar part must be null

$$\nabla \cdot J = \partial_0\rho + \boldsymbol{\nabla} \cdot \mathbf{J} = 0. \tag{4.15}$$

From (4.15) we obtain the usual charge continuity equation.

4.1 The Vector Potential

Being $\nabla \wedge F = 0$, F can be expressed as the derivative of a vector potential A:

$$F = \nabla A = \nabla \cdot A + \nabla \wedge A. \tag{4.16}$$

Since F is a bivector, it follows that:

$$F = \nabla \wedge A \tag{4.17}$$

$$\nabla \cdot A = 0. \tag{4.18}$$

Therefore, if we wish we can write Maxwell equations in the form:

$$\nabla^2 A = J. \tag{4.19}$$

There is some freedom in the choice of A, since we can always add the gradient of a scalar field λ such that $\nabla^2 \lambda = 0$:

$$\nabla(A + \nabla\lambda) = \nabla A + \nabla^2 \lambda = \nabla A. \tag{4.20}$$

This freedom to alter A is called for historical reason *gauge freedom*.

If we wish, we can solve $\mathbf{E}$ and $\mathbf{B}$ in terms of vector potential. Being:

$$A\gamma_0 = \phi + \mathbf{A}, \tag{4.21}$$

then

$$\begin{aligned} F &= \nabla A \\ &= \nabla \gamma_0 \gamma_0 A \\ &= (\partial_0 - \boldsymbol{\nabla})(\phi - \mathbf{A}) \\ &= -\partial_0 \mathbf{A} - \boldsymbol{\nabla}\phi + \boldsymbol{\nabla} \wedge \mathbf{A}. \end{aligned} \tag{4.22}$$

Recalling that $F = \mathbf{E} + I\mathbf{B}$ and equating the same grade terms on the left and right side:

$$\begin{aligned} \mathbf{E} &= -(\partial_0 \mathbf{A} + \boldsymbol{\nabla}\phi) \\ \mathbf{B} &= \nabla \times \mathbf{A} \end{aligned} \tag{4.23}$$

4.2 Electromagnetic Field Strength

The Faraday bivector F is also known as *electromagnetic field strength*. F is a covariant spacetime bivector. Its components in the $\{\gamma^\mu\}$ frame are also the components of a rank 2 antisymmetric tensor:

$$F^{\mu\nu} = \gamma^\nu \cdot (\gamma^\mu \cdot F) = (\gamma^\nu \wedge \gamma^\mu) \cdot F. \tag{4.24}$$

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix} \tag{4.25}$$

which is the form usually presented in relativistic electrodynamics books.

Suppose we have another observer moving at a velocity $v = R\gamma_0\tilde{R}$. If we build a frame for the second observer with basis vectors:

$$\gamma'_\mu = R\gamma_\mu\tilde{R} \tag{4.26}$$

it will measure the components of the electric field to be

$$E_i = (\gamma'_i\gamma'_0)\cdot F = (R\boldsymbol{\sigma}_i\tilde{R})\cdot F = \boldsymbol{\sigma_i}\cdot(\tilde{R}FR) \tag{4.27}$$

Again, we see the effect of a Lorentz transformation as a generalized rotation in the spacetime of the Faraday bivector F. It is important to notice that bivectors and vectors are subject to the same rotor transformation laws.

Now suppose that the Faraday bivector F is measured in the same point by two observers, one who has a 4-velocity γ_0 while the other has a relative velocity $\mathbf{v}$ in the γ_0 frame. As we have seen in the previous chapter, this means that the second observer has a 4-velocity v:

$$v = R\gamma_0\tilde{R} \tag{4.28}$$

where $R = e^{\alpha\hat{\mathbf{v}}/2}$, and $\mathbf{v} = \tanh(\alpha)\hat{\mathbf{v}}$. In order to find the components of $\tilde{R}FR$ in the $\{\gamma_\mu\}$ frame, we decompose the Faraday bivector into terms parallel and perpendicular to $\mathbf{v}$:

$$\mathbf{v}F_\parallel = F_\parallel\mathbf{v} \qquad \mathbf{v}F_\perp = -F_\perp\mathbf{v}. \tag{4.29}$$

The parallel terms are unchanged by the transformation, whereas the perpendicular terms transform as follows:

$$\tilde{R}F_\perp R = e^{-\alpha\hat{\mathbf{v}}}F_\perp = \gamma(1-\mathbf{v})F_\perp, \tag{4.30}$$

where γ is the Lorentz factor. From (4.30) it is straightforward to get the following transformation laws for the electric and magnetic field:

$$\begin{aligned}\mathbf{E}'_\perp &= \gamma(\mathbf{E}+\mathbf{v}\times\mathbf{B})_\perp,\\ \mathbf{B}'_\perp &= \gamma(\mathbf{E}-\mathbf{v}\times\mathbf{B})_\perp.\end{aligned} \tag{4.31}$$

Moreover, the invariants of the electromagnetic field are given by F^2:

$$F^2 = F\cdot F + F\wedge F. \tag{4.32}$$

It is clear the F^2 is Lorentz-invariant:

$$(\tilde{R}FR)(\tilde{R}FR) = \tilde{R}FFR = F^2. \tag{4.33}$$

The square of a bivector may contain only scalar and pseudoscalar terms, therefore F^2 can be written as:

$$F^2 =< F^2 > + < F^2 >_4 = a_0 + Ia_4. \tag{4.34}$$

Both the scalar a_0 and the pseudoscalar Ia_4 are Lorentz invariants. If we wish, after having chosen a particular frame γ_0, we can write:

$$\begin{aligned} \alpha &=< (\mathbf{E} + I\mathbf{B})(\mathbf{E} + I\mathbf{B}) >= \mathbf{E}^2 - \mathbf{B}^2, \\ \beta &= - < I(\mathbf{E} + I\mathbf{B})(\mathbf{E} + I\mathbf{B}) >= 2\mathbf{E} \cdot \mathbf{B}. \end{aligned} \tag{4.35}$$

The first one is the *Lagrangian density of the electromagnetic field*, while the latter is somewhat less commonly used: nonetheless, it is interesting to notice that $\mathbf{E} \cdot \mathbf{B}$ is invariant under not only rotations, but under every Lorentz transformations.

For F to be decomposed in $F = \mathbf{E} + I\mathbf{B}$, we had to choose a particular timelike direction γ_0. There is another way to decompose F using only invariants of the field, and we are going to show it here only for the case $F^2 \neq 0$. First, we note that F can be written as follows:

$$F = fe^{I\phi}, \tag{4.36}$$

where ϕ is a scalar and

$$f = \mathbf{e} + I\mathbf{b}, \tag{4.37}$$

with $\mathbf{e} \cdot \mathbf{b} = 0$. When $F \neq 0$, both f and ϕ are determined by the scalar F^2:

$$F^2 = f^2 e^{2I\phi} = (\mathbf{e}^2 - \mathbf{b}^2)e^{2I\phi}. \tag{4.38}$$

In this last expression, we see that F^2 can be seen as a scalar f^2 which is taken through 2ϕ by a duality rotation, in order to get the scalar and pseudoscalar terms of F^2.

Let us express f^2 and ϕ in terms of the invariants $\mathbf{E}^2 - \mathbf{B}^2$ and $\mathbf{E} \cdot \mathbf{B}$. We start by evaluating F^{*2}

$$F^{*2} = F^{2*} = f^2 e^{-2I\phi} = \mathbf{E}^2 - \mathbf{B}^2 - 2I\mathbf{E} \cdot \mathbf{B}. \tag{4.39}$$

Since

$$F^{*2}F^2 = (F^*F)^2 = f^4 = (\mathbf{E}^2 - \mathbf{B}^2)^2 + 4(\mathbf{E} \cdot \mathbf{B})^2, \tag{4.40}$$

we can write f^2:

$$f^2 = \pm[(\mathbf{E}^2 - \mathbf{B}^2)^2 + 4(\mathbf{E} \cdot \mathbf{B})^2]^{1/2}. \tag{4.41}$$

There is also another expression for f^2, that we can find by noticing that:

$$F^*F = -(\mathbf{E}^2 + \mathbf{B}^2) + 2\mathbf{E} \times \mathbf{B}. \tag{4.42}$$

Since $\mathbf{E} \times \mathbf{B}$ anticommutes with both $\mathbf{E}$ and $\mathbf{B}$, we find:

$$f^4 = F^*F^*FF = (F^*F)(F^*F)^* = (\mathbf{E}^2 + \mathbf{B}^2)^2 - 4(\mathbf{E} \times \mathbf{B})^2, \tag{4.43}$$

therefore

$$f^2 = \pm[(\mathbf{E}^2 + \mathbf{B}^2)^2 + 2(\mathbf{E} \times \mathbf{B})^4]^{1/2}. \tag{4.44}$$

The expression for ϕ is pretty simple, since we know that $\tan(\phi)$ is the ratio between the pseudoscalar and the scalar terms:

$$\phi = \arctan \frac{2\mathbf{E} \cdot \mathbf{B}}{\mathbf{E}^2 - \mathbf{B}^2}. \tag{4.45}$$

The previous expression works only if $F^2 \neq 0$ and only to an additive multiple of π. Usually ϕ is called the complexion of the Maxwell field; David Hestenes was the first to point out that ϕ can be interpreted as the physical *phase* of the electromagnetic field [8].

4.3 Free Fields

The source-free Maxwell equation

$$\nabla F = 0 \tag{4.46}$$

is the same equation for a neutrino field, so it is unsurprising that we can describe the polarization of a photon in the same manner we describe the polarization of a neutrino. A solution for (4.46) is the plane wave

$$F(x) = f e^{Ik \cdot x} \tag{4.47}$$

where $x = x^\mu \gamma_\mu$, is the position in the Minkowski spacetime, $k = k^\mu \gamma_\mu$ is the wavevector and f is a constant bivector. If we choose a particular spacetime direction γ_0, we obtain from (4.46) the expression:

$$\gamma_0 \nabla F = (\partial_0 + \boldsymbol{\nabla})F = 0. \tag{4.48}$$

We now recall that $k\gamma_0 = \omega + \mathbf{k}$. If we now put the solution (4.47) in (4.48), we obtain:

$$\mathbf{k} f = \omega f, \tag{4.49}$$

and, multiplying this last expression by $k\gamma_0$ we obtain the relation between ω and $\mathbf{k}$ we already found in section (3.6):

$$\omega = \pm|\mathbf{k}|. \tag{4.50}$$

The two solutions correspond to the left and right circularly polarised photons, i.e. the helicity. For the case of neutrinos, we can view $\omega = |\mathbf{k}|$ as the neutrino and $\omega = -|\mathbf{k}|$ as the anti-neutrino. We can therefore interpret the operation of space conjugation with that of particle-antiparticle conjugation. So, from (4.49) we obtain:

$$-\mathbf{k}f^* = \omega f^*, \tag{4.51}$$

In order to limit the number of solutions for the Dirac equations, we must impose a few conditions on the field. These conditions are analogous to the following condition for the electromagnetic field:

$$F = \bar{F} \tag{4.52}$$

but this condition is trivially satisfied for a bivector. We want to take advantage from the fact that f is a bivector to show some properties of (4.49). As in the previous section, we write:

$$f = \mathbf{e} + I\mathbf{b} \tag{4.53}$$

therefore, it is possible to split (4.49) in two terms:

$$\begin{aligned} \omega\mathbf{e} &= I\mathbf{k}\mathbf{b}, \\ \omega\mathbf{b} &= -I\mathbf{k}\mathbf{e} \end{aligned} \tag{4.54}$$

From these latter equations, we get the conditions:

$$\begin{aligned} &\mathbf{k} = \omega\hat{\mathbf{e}} \times \hat{\mathbf{b}}, \\ &\mathbf{k}\cdot\mathbf{e} = \mathbf{k}\cdot\mathbf{b} = 0, \\ &\mathbf{e}\cdot\mathbf{b} = 0, \\ &\mathbf{e}^2 = \mathbf{b}^2. \end{aligned} \tag{4.55}$$

The first three conditions show the orthogonality of the vectors $\mathbf{k}$, $\mathbf{e}$ and $\mathbf{b}$. It is noteworthy that condition (3) and (4) are summarised by

$$f^2 = 0, \tag{4.56}$$

which in turn implies that

$$F^2 = 0. \tag{4.57}$$

The latter expression is an invariant condition that F is a circularly polarised field.

We conclude by adding a geometrical interpretation to the plane wave solution. We can see $e^{Ik\cdot x}$ as a duality rotation operator action on f, which transforms $\mathbf{e}$ in a bivector and $I\mathbf{b}$ in a vector. We can picture the electric and magnetic vector spinning around the momentum vector $\mathbf{k}$: this effect is rather due by a duality transformation than an actual spatial rotation.

Chapter 5

Conclusions

This work tried to give a brief but somewhat comprehensive overview of the classical and relativistic applications of Clifford algebra.

In classical mechanics, we saw how different instruments such as Pauli matrices and quaternions, are actually different representations of a bigger structure, and how it is easy to give them geometrical meaning. Nonetheless, we found that although in classical mechanics Clifford algebra provides additional geometrical clarity and efficiency, it is not essential for a synthetic treatment of three-dimensional physics.

This is not the case for spacetime physics: we cannot define a cross product in a 4-dimensional space (since, as we have seen, the Hodge dual of a bivector is also a bivector and not a common vector). Therefore, most texts revert to a more basic approach, which involves the components of 4-vectors and Lorentz-transform matrices. Such an approach tends to estrange from the geometry of spacetime, where on the contrary Clifford algebra (and exterior forms, which are actually a subset of Clifford algebra [7]) does not.

One of the most astounding results of Clifford spacetime algebra is its ability to encode a lot of information in its structure: in the last chapter, we saw how all Maxwell equations can be written as $\nabla F = J$, and, by doing elementary analysis, we were able to get from this formulation a lot of properties of electromagnetic fields. The result we got are usually difficult to obtain and visualize using tensor or vector analysis.

It should also be pointed out that in the approach of these thesis there have not been any use of matrices, if not for referencing orthodox literature on the subject we discussed. We wanted to disclaim the erroneous belief that matrices are crucial to understanding the properties of Clifford algebras. Of course, since Clifford algebras are associative algebras, they always have a matrix representation [9]: nonetheless they add little to understanding the basic properties of the algebras. Moreover, the size of matrices increases exponentially with the dimension of the space, making computations slower.

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