Exercises QFT II — 2018/2019

Problem Sheet 1

Problem 1: Path integrals in Quantum mechanics

The transition amplitude between a state $|q_a\rangle$ at $t = 0$ and $|q_b\rangle$ at $t = T$ can be expressed as

$$\langle q_b|e^{-iHT}|q_a\rangle \;=\; \int \mathcal{D}q\, e^{iS[q]}$$

where the integral is done over all the possible trajectories connecting the points $q(0) = q_a$ and $q(T) = q_b$, and $S[q] \;=\; \int_0^T dt\, \mathcal{L}[q, \dot{q}]$. In this problem we set $\hbar = 1$.

We are now going to determine, in few steps, the amplitude in the case of the harmonic oscillator:

$$\mathcal{L} \;=\; \frac{m}{2}\,\dot{q}^2 - \frac{m\,\omega^2}{2}\,q^2.$$

1. Recall that the classical trajectory $q_c(t)$ is found by minimizing the action:

$$-\frac{\delta S}{\delta q(t)} \;=\; m\ddot{q}(t) + m\omega^2\, q(t) \;=\; 0$$

   and imposing the boundary conditions $q(0) = q_a$ and $q(T) = q_b$. An arbitrary trajectory $q(t)$ can then be decomposed as $q(t) \equiv q_c(t) + y(t)$ with the boundary conditions $y(0) = y(T) = 0$.

2. The following expression for the action is exact (why are there no higher order terms?)

$$S[q] \;=\; S[q_c] + \int_0^T dt\, \frac{\delta S[q_c]}{\delta q(t)}\, y(t) + \frac{1}{2}\int_0^T\int_0^T dt dt'\, \frac{\delta^2 S[q_c]}{\delta q(t)\delta q(t')}\, y(t) y(t')\,.$$

   Show that we can write

$$\frac{\delta^2 S[q_c]}{\delta q(t)\delta q(t')} \;=\; -m\,\frac{d^2}{dt^2}\,\delta(t-t') - m\,\omega^2\,\delta(t-t')$$

   so that $S[q] \;=\; S[q_c] + \frac{m}{2}\int_0^T \left(\dot{y}^2 - \omega^2\, y^2\right) \;\equiv\; S[q_c] + S[y]$. Then our initial amplitude reads

$$\int \mathcal{D}q\, e^{i\,S[q]} \;=\; e^{i\,S[q_c]} \int \mathcal{D}y\, e^{i\,S[y]}$$

3. Show that

$$S[q_c] = \frac{m\omega}{2\sin\omega T}\, \left[(q_b^2 + q_a^2)\,\cos\omega T - 2q_a q_b\right]\,.$$

4. Let us introduce a basis of functions satisfying our boundary conditions:

$$y_n(t) \;=\; \mathcal{C}_n \, \sin\left(\frac{n\pi t}{T}\right)$$

such that on the interval $[0,T]$: $\int_0^T y_n y_m \;=\; \delta_{nm}$. Determine the constant $\mathcal{C}_n$. We can then expand:

$$y(t) \;=\; \sum_{n=1}^{\infty} a_n y_n(t)\,.$$

where the coefficients a_n are constant. Show that $S[y] \;=\; \frac{m}{2}\sum_1^\infty \lambda_n\, a_n^2$ and find the value of λ_n.

5. The integral measure can be expressed as (accept it as a postulate, but try to think about it):

$$\mathcal{D}y \;=\; J\prod_{n=1}^{\infty} da_n\;, \qquad\qquad \text{for some constant } J.$$

Knowing this, show that $F_\omega(T) \;\equiv\; \int \mathcal{D}y\, e^{iS[y]} \;=\; J\prod_{n=1}^{\infty}\left(\frac{m}{2\pi i}\,\lambda_n\right)^{-\frac{1}{2}}$

6. We know the *exact* value of $F_\omega(T)$ for the case of free fields, $\omega = 0$: recall indeed that in this case $F_0(T) \;=\; \left(\frac{m}{2\pi i T}\right)^{\frac{1}{2}}$. On the other hand, one may also want to calculate $F_0(T)$ by the same procedure we developed until now: show that the λ_n coefficients, when $\omega = 0$, read $\lambda_n^{(0)} \;=\; \frac{n^2\,\pi^2}{T^2}$.

7. Then we can write

$$\frac{F_\omega(T)}{F_0(T)} \;=\; \prod_{n=1}^{\infty}\left(\frac{\lambda_n}{\lambda_n^{(0)}}\right)^{-\frac{1}{2}} \;=\; \prod_{n=1}^{\infty}\left(1-\frac{\omega^2 T^2}{\pi^2\, n^2}\right)^{-\frac{1}{2}}\,.$$

Deduce from this that

$$F_\omega(T) \;=\; \left(\frac{m\,\omega}{2\pi\, i\,\sin\left(\omega T\right)}\right)^{\frac{1}{2}}$$

and, finally, collect everything and write up the result for the transition amplitude!

Problem 2: D_F^{-1}

Consider the operator

$$iD_F^{-1}(x,y) = (-\Box_y - m^2 + i\epsilon)\delta^4(x-y)\,. \tag{1}$$

It acts on functions $f:\mathbb{R}^4 \to \mathbb{R}$ as $iD_F^{-1}[f](y) = \int d^4x f(x) iD_F^{-1}(x,y)$.

Show that D_F^{-1} is actually the inverse of D_F, i.e. that

$$D_F^{-1}\, D_F\, \phi = \phi \qquad\qquad \forall\phi(x)\,. \tag{2}$$