Exercises QFT II — 2018/2019

Problem Sheet 2

Problem 3: Quantum Harmonic Oscillator

Consider the harmonic oscillator with Lagrangian and Hamiltonian

$$\mathcal{L} = \frac{1}{2}\dot{q}^2 - \frac{\omega^2}{2}q^2\,, \qquad H = \frac{p^2}{2} + \frac{\omega^2}{2}q^2\,.$$

Canonical quantization promotes q, p to operators Q, P statisfying the commutation relations $[Q, P] = i$ (we set $\hbar = 1$). We will work in Heisenberg picture. Let us consider a set of eigenstates $|q, \pm\infty\rangle$ of the operator $Q(t)$ at $t \to \pm\infty$):

$$Q(\pm\infty)|q, \pm\infty\rangle = q\,|q, \pm\infty\rangle\,.$$

To construct the energy eigenstate it is more useful to define the annihilation and creation operators:

$$a \equiv \sqrt{\frac{\omega}{2}}Q + i\frac{1}{\sqrt{2\omega}}P \qquad a^\dagger \equiv \sqrt{\frac{\omega}{2}}Q - i\frac{1}{\sqrt{2\omega}}P \tag{1}$$

with commutation relation $[a, a^\dagger] = 1$. The ground state of the Energy is the state $|\Omega\rangle$ (that we will call the *vacuum*) satisfying

$$a\,|\Omega\rangle = 0\,.$$

When we work in Heisenberg picture, the operator a takes a time dependence: $a(t)$. We can then define the vacuum state at $t \to \pm\infty$ as

$$\lim_{t\to\pm\infty} a(t)\,|\Omega, \pm\infty\rangle = 0\,. \tag{2}$$

Consider the quantity

$$G(q, t) \equiv \langle q, t|\,\Omega, t\rangle \tag{3}$$

1. Show that $G(q, t)$ does not depend on t. Hence from now on $G(q, t) = G(q)$.
2. By using eq.(2) and the representation of P in q-space ($\langle q|P|\psi\rangle = -i\frac{d}{dq}\langle q|\psi\rangle$), show that $G(q)$ satisfies the following differential equation

$$\frac{d}{dq}G(q) = -\omega\,q\,G(q)\,. \tag{4}$$

3. Show that its solution is

$$G(q) = e^{-\frac{\omega}{2}q^2}\,. \tag{5}$$

Consider now the vacuum amplitude

$$\langle \Omega, +\infty\,|\,\Omega, -\infty\rangle = \int dq' \int dq\,\langle \Omega, +\infty\,|q', +\infty\rangle\,\langle q', +\infty\,|\,q, -\infty\rangle\,\langle q, -\infty\,|\,\Omega, -\infty\rangle \tag{6}$$

with $\langle q', +\infty\,|\,q, -\infty\rangle$ computed in *Exercise 1*, $\langle \Omega, +\infty\,|q', +\infty\rangle = G(q')$ and $\langle q, -\infty\,|\,\Omega, -\infty\rangle = G(q)$.

4. Collect everything together to show that

$$\langle \Omega, +\infty \,|\, \Omega, -\infty \rangle = \int \mathcal{D}q \, e^{i \int dt \, \mathcal{L} - \frac{\epsilon}{2} \int dt \, \omega \, q(t)^2} \tag{7}$$

where the path integral is unconstrained (i.e. no fixed boundary conditions at $t \to \pm\infty$). [Remember that $f(+\infty) + f(-\infty) = \lim_{\epsilon \to 0^+} \epsilon \int_{-\infty}^{+\infty} dt f(t) e^{-\epsilon |t|}$].

Problem 4: Forced Harmonic Oscillator

Define

$$Z[F] \equiv \langle \Omega, +\infty \,|\, \Omega, -\infty \rangle_F = \int \mathcal{D}q \, e^{i \int dt \, \mathcal{L} - \frac{\epsilon}{2} \int dt \, \omega \, q(t)^2 + i \int dt \, F(t) \, q(t)} \tag{8}$$

i.e. the vacuum amplitude in presence of an external force F.

1. Compute the classical equation of motion for $q(t)$ when $\mathcal{L} = \frac{1}{2}\dot{q}^2 - \frac{\omega^2}{2} q^2 + F \, q$ and justify why we call F force.

2. Show that the T-ordered correlators for the Harmonic Oscillator can be derived from $Z[F]$ by

$$\langle \Omega, +\infty \,|\, \mathrm{T} \, q(t_1) ... q(t_n) \,|\Omega, -\infty \rangle = (-i)^n \left. \frac{\delta^n Z[F]}{\delta F(t_1) ... \delta F(t_n)} \right|_{F=0} \tag{9}$$

3. By using the Fourier transforms of $q(t)$ and $F(t)$, show that

$$Z[F] = Z[0] \, e^{-\frac{i}{2} \int dt_1 dt_2 F(t_1) D(t_1 - t_2) F(t_2)} \tag{10}$$

with

$$D(t_1 - t_2) = \int \frac{dE}{2\pi} \frac{1}{E^2 - \omega^2 + i \, \epsilon \, \omega} \, e^{i \, E \, (t_1 - t_2)} \tag{11}$$

[Suggestion: Use the same procedure we did in *Lecture 4* to compute $Z[J]$ in a free scalar theory.]

4. Compute the four point function

$$\langle \Omega, +\infty \,|\, \mathrm{T} \, q(t_1) q(t_2) q(t_3) q(t_4) \,|\Omega, -\infty \rangle \, . \tag{12}$$

[Use eq(9).]