Exercises QFT II — 2018/2019

Problem Sheet 3

Problem 5: Effective action and Green's functions

Consider the 1PI effective action

$$\Gamma[\psi] \equiv W[J] - \int d^4x\, J(x)\, \psi(x) \qquad\qquad (\, e^{iW[J]} = Z[J]\,)\,, \tag{1}$$

where

$$\psi(x) \equiv \langle\Omega|\, \phi(x)\, |\Omega\rangle_J = \frac{\delta W[J]}{\delta J}\,. \tag{2}$$

1. Verify that the ψ given by (2) satisfies the equation $\frac{\delta\Gamma}{\delta\psi(x)} = -J(x)$.

Consider the free scalar theory given by

$$S = \int d^4x \left(\frac{1}{2}\partial_\mu\phi(x)\partial^\mu\phi(x) - \frac{m^2}{2}\phi(x)^2\right)\,. \tag{3}$$

2. Compute $\Gamma[\psi]$ for this simple case. Remember that for a free scalar theory

$$W[J] = -\frac{1}{2}\int d^4x\, d^4y\, J(x)\Delta_F(x-y)J(y) \qquad \text{and} \qquad \Delta_F(x-y) = \int \frac{d^4p}{(2\pi)^4}\frac{e^{-ip\cdot(x-y)}}{p^2 - m^2 + i\epsilon}\,.$$

Define

$$\Gamma^{(n)}(x_1, ..., x_n) \equiv \left.\frac{\delta^n\Gamma[\psi]}{\delta\psi(x_1)...\delta\psi(x_n)}\right|_{\psi=0}\,. \tag{4}$$

3. Compute the Fourier transform $\tilde{\Gamma}^{(2)}(p_1, p_2)$ of $\Gamma^{(2)}(x_1, x_2)$, where

$$\tilde{\Gamma}^{(n)}(p_1, ..., p_n) = \int \frac{d^4x_1}{(2\pi)^2}...\int\frac{d^4x_n}{(2\pi)^2}e^{-i(p_1\cdot x_1 + ... + p_n\cdot x_n)}\Gamma^{(n)}(x_1, ..., x_n)\,. \tag{5}$$

4. Show that

$$\int d^4y\, G^{(2)}(x,y)\Gamma^{(2)}(y,z) = \alpha\, \delta^4(x-z)$$

and find the value of α.

Let us now show that the 2-point Green's function depends only on the difference of the point positions, i.e.

$$\langle\Omega|\, T\, \hat{\phi}(x_1)\hat{\phi}(x_2)\, |\Omega\rangle = G^{(2)}(x_1 - x_2) \tag{6}$$

This follows form the invariance of the vacuum state under space-time translation, i.e. $\hat{P}_\mu|\Omega\rangle = 0$.

5. Using this property of $|\Omega\rangle$ and the transformation low of the field $\hat{\phi}(x)$ under translation, show that

$$\langle\Omega|\, T\, \hat{\phi}(x_1)\hat{\phi}(x_2)\, |\Omega\rangle = \langle\Omega|\, T\, \hat{\phi}(x_1 + v)\hat{\phi}(x_2 + v)\, |\Omega\rangle$$

where v is a constant space-time shift. Why does this imply (6)?

Problem 6: Phase space Path Integral

We have seen in Lecture 1, that

$$\langle q', t_f | q, t_i \rangle \;=\; \lim_{\delta t \to 0} \prod_{k=1}^{n-1} \left\{ \int dq_i \left(\frac{m}{2\pi i \hbar \delta t} \right)^{1/2} \right\} e^{i \sum_{k=0}^{n-1} L[q_k, \dot{q}_k]/\hbar} \equiv \int \mathcal{D}q \, e^{i/\hbar \int_{t_i}^{t_f} L[q,\dot{q}] dt} \tag{7}$$

(with proper boundary conditions for $q(t)$). To derive this formula we assumed to have a system with $H = \frac{p^2}{2m} + V(q)$.

Consider instead an Hamiltonian of the form $H = \frac{1}{2} p^2 v(q)$.

- Show that the classical Lagrangian is $L = \frac{\dot{q}^2}{2v(q)}$.
- Show that

$$\langle q', t_f | q, t_i \rangle \;\neq\; \int \mathcal{D}q \, e^{i/\hbar \int_{t_i}^{t_f} L[q,\dot{q}] dt}.$$

  where

$$\langle q', t_f | q, t_i \rangle \;=\; \int \mathcal{D}q \, \mathcal{D}p e^{i/\hbar \int_{t_i}^{t_f} dt \{p\dot{q} - H\}} \; . \tag{8}$$