Teor. Identità di Cauchy

Sia $G \subset \mathbb{C}$ aperto limitato con front. reg. a tratti. Sia $f \in C^1(\overline{G}) \cap H(G)$.

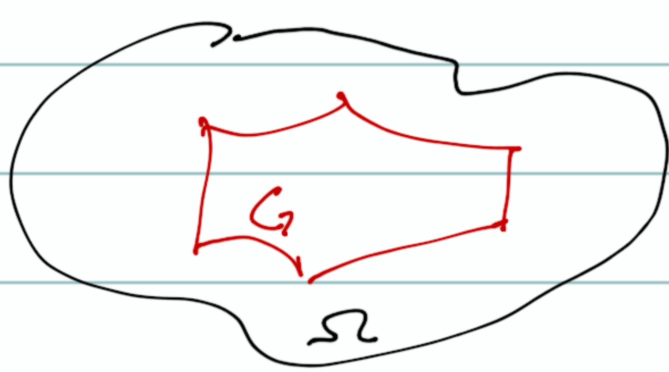
Allora

$$\int_{\partial G} f(z)\,dz = 0$$

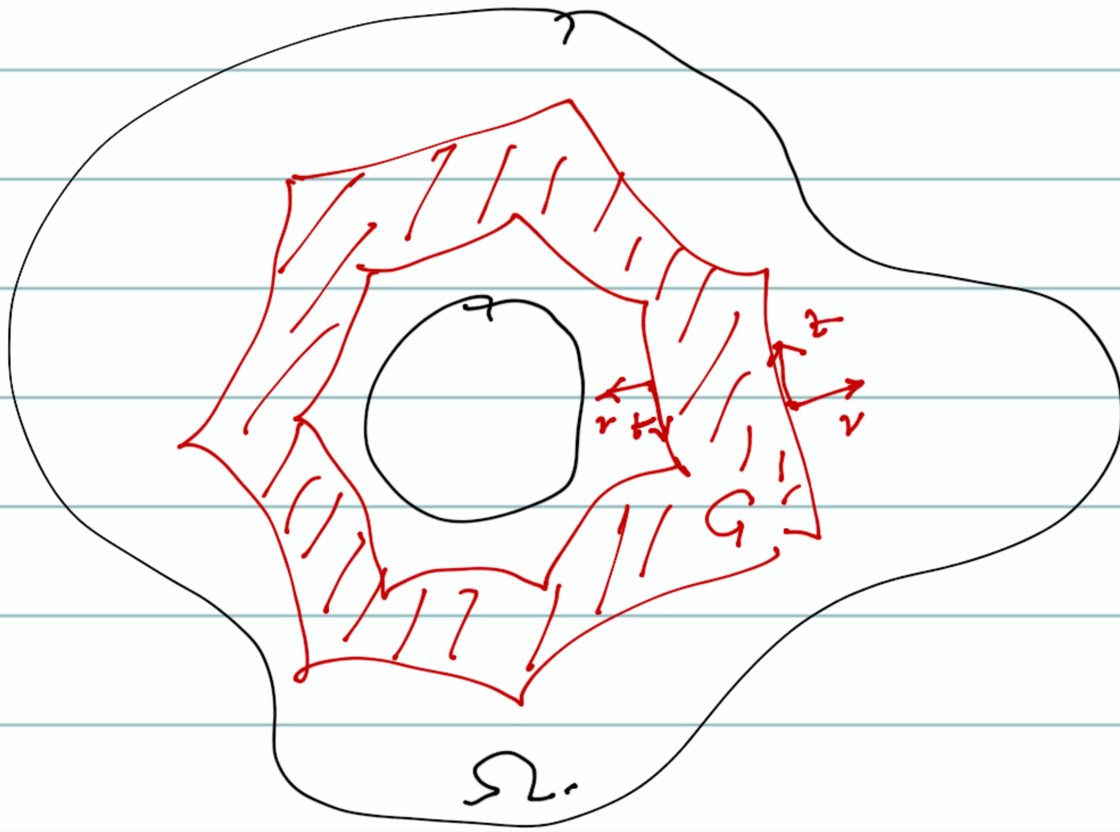
Dim

$$\int_{\partial G} f(z)\,dz = 2i \int_G \underbrace{\partial_{\bar z} f(z)}_{=0}\,dx\,dy = 0 \qquad \square$$

Oss. Se $f \in H(\Omega)$, e G è aperto con front. reg. a tratti t.c. $\overline{G} \subset \Omega$



$$\int_{\partial G} f(z)\,dz = 0$$



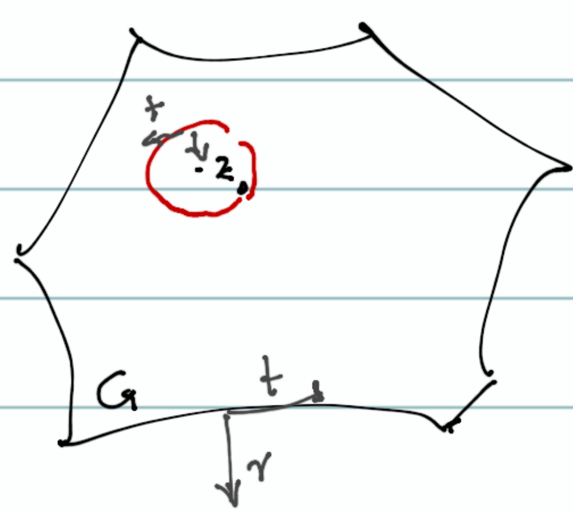
Formula di Cauchy.

Sia G come prima. Sia $f \in C^1(\overline{G}) \cap H(G)$.

$\forall\ z_0 \in G$

$$f(z_0) = \frac{1}{2\pi i} \int_{\partial G} \frac{f(z)}{z - z_0}\, dz .$$

Dim.



$\exists\ r > 0$ t.c. $\overline{B_r(z_0)} \subset G$

Sia $\varepsilon > 0$ t.c. $\varepsilon \leq r$.

Poniamo $G_\varepsilon = G \setminus \overline{B_\varepsilon(z_0)}$. G_ε ha front. reg a tratti

$$\partial G_\varepsilon = \partial G - \partial B_\varepsilon(z_0).$$

Poniamo

$$\underline{g(z)} = \boxed{\frac{f(z) - f(z_0)}{z - z_0}} \in H(G \setminus \{z_0\})$$

$$\int_{\partial G_\varepsilon} g(z)\,dz = 0 \quad \text{per l'Id. di Cauchy.}$$

$$\int_{\partial G} g(z)\,dz - \int_{\partial B_\varepsilon(z_0)} g(z)\,dz = 0$$

$$\boxed{\int_{\partial G} \frac{f(z) - f(z_0)}{z - z_0}\,dz = \int_{\partial B_\varepsilon(z_0)} \frac{f(z) - f(z_0)}{z - z_0}\,dz}$$

$$\int_{\partial G_\varepsilon} \frac{f(z_0)}{z - z_0}\,dz = 0$$

$$\int_{\partial G} \frac{f(z_0)}{z - z_0}\,dz = \int_{\partial B_\varepsilon(z_0)} \frac{f(z_0)}{z - z_0}\,dz =$$

$$= f(z_0) \int_{\partial B_\varepsilon(z_0)} \frac{dz}{z - z_0} = 2\pi i\, f(z_0)$$

$$\boxed{\int_{\partial G} \frac{f(z)}{z - z_0}\, dz = 2\pi i\, f(z_0)} + \underbrace{\int_{\partial B_\varepsilon(z_0)} \left(\frac{f(z) - f(z_0)}{z - z_0} \right) dz}_{\to 0}$$

$$\int_{\partial B_\varepsilon(z_0)} \left(\frac{f(z) - f(z_0)}{z - z_0} \right) dz = O(\varepsilon) \qquad \varepsilon \to 0+$$

$$f(z) - f(z_0) = f'(z_0)(z - z_0) + o(|z - z_0|)$$

$$\frac{f(z) - f(z_0)}{z - z_0} = f'(z_0) + o(1) \qquad z \to z_0$$

So $\partial B_\varepsilon(z_0)$ $\quad \dfrac{f(z) - f(z_0)}{z - z_0} = f'(z_0) + o(1) \qquad \varepsilon \to 0.$

For $\varepsilon \in (0, r]$ $\exists\, M > 0$ s.t.

$$\left| \frac{f(z) - f(z_0)}{z - z_0} \right| \leq M$$

$$\left| \int_{\partial B_\varepsilon(z_0)} \frac{f(z)-f(z_0)}{z-z_0} \, dz \right| \leq M \, L(\partial B_\varepsilon) =$$

$$= M \, 2\pi\varepsilon \to 0 \quad \varepsilon \to 0.$$

Quindi:

$$\int_{\partial G} \frac{f(z)}{z-z_0} \, dz = \underline{2\pi i \, f(z_0)} \qquad \square$$

$\forall z \in G$

$$\boxed{f(z) = \frac{1}{2\pi i} \int_{\partial G} \frac{f(\zeta)}{\zeta - z} \, d\zeta}$$

ζ = zeta greco "zita"

$\zeta = \xi + i\eta$, $\xi = \mathrm{Re}\,\zeta$

$\eta = \mathrm{Im}\,\zeta$

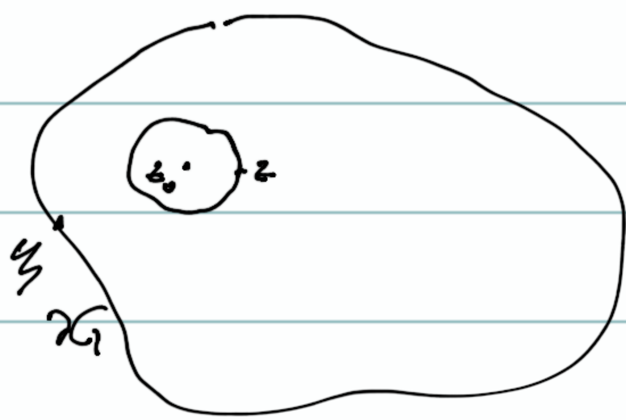
Corollario Sia G come prima. $f \in C^1(\bar{G}) \cap H(G)$.

Allora f è svilupp. in serie di potenze.

$$\boxed{f(z) = \frac{1}{2\pi i} \int_{\partial G} \frac{f(\xi)}{\xi - z}\, d\xi}$$

Sia $z_0 \in G$ e sia $z \sim z_0$

$$f(z) = \frac{1}{2\pi i} \int_{\partial G} \frac{f(\xi)}{(\xi - z_0) - (z - z_0)}\, d\xi$$



$$f(z) = \frac{1}{2\pi i} \int_{\partial G} \frac{f(\xi)}{\xi - z_0} \left(\frac{1}{1 - \frac{z - z_0}{\xi - z_0}} \right) d\xi =$$

$$= \frac{1}{2\pi i} \int_{\partial G} \frac{f(\xi)}{\xi - z_0} \sum_{n=0}^{\infty} \left(\frac{z - z_0}{\xi - z_0} \right)^n d\xi$$

$\perp$

per ogni $z \in B_r(z_0)$, con $r > 0$ t.c. $B_r(z_0) \subset G$.

$$\boxed{f(z) = \sum_{n=0}^{\infty} \left[\frac{1}{2\pi i} \int_{\partial G} \frac{f(\xi)}{(\xi - z_0)^{n+1}} \, d\xi \right] (z - z_0)^n} \quad (*)$$

questo è lo svil. in serie di Taylor di f con centro z_0

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n \qquad (**)$$

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_{\partial G} \frac{f(\xi)}{(\xi - z_0)^{n+1}} \, d\xi$$

$(*)$ Questa formula garantisce che f è derivabile infinite volte in senso complesso e che tutte le deriv. sono olomorfe.

In gen: Se f è svil. in serie di potenze

$$f(z) = \sum^{\infty} a_n (z - z_0)^n$$

$n=0$

$$f(z_0) = a_0 \quad , \quad f'(z) = \sum_{n=0}^{\infty} n a_n (z-z_0)^{n-1}$$

$$f'(z_0) = a_1$$

$$f^{(n)}(z_0) = n!\, a_n$$

<u>Lemma</u> Sia $f \in H(\Omega)$, sia $z_0 \in \Omega$, sia $R > 0$
t.c. $\overline{B_R(z_0)} \subset \Omega$.
Posto $M = \max_{\partial B_R(z_0)} |f|$ si ricava

$$|f^{(k)}(z_0)| \leq \frac{M\, k!}{R^k}.$$

<u>Dim.</u>

$$f^{(k)}(z_0) = \frac{k!}{2\pi i} \int_{\partial B_R(z_0)} \frac{f(\xi)}{(\xi - z_0)^{k+1}} \, d\xi$$

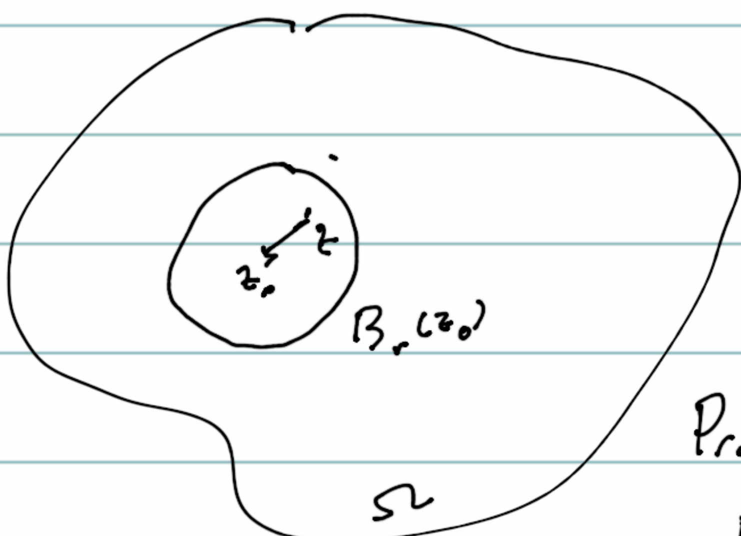
$$|f^{(k)}(z_0)| \leq \frac{k!}{2\pi} \frac{M}{R^{k+1}} 2\pi R = \frac{M\, k!}{R^k} \qquad \square$$

<u>Teorema di Morera</u> Sia $f \in C(\Omega; \mathbb{C})$ su un aperto $\Omega \subset \mathbb{C}$, tale che per ogni dominio limitato $D \subset \overline{D} \subset \Omega$ valga (∂D reg. a tratti!)

$$\int_{\partial D} f(z)\, dz = 0$$

Allora $f \in H(\Omega)$.

<u>Dim.</u> Sia $z_0 \in \Omega$ e sia $r > 0$ t.c. $\overline{B_r(z_0)} \subset \Omega$.



Poniamo

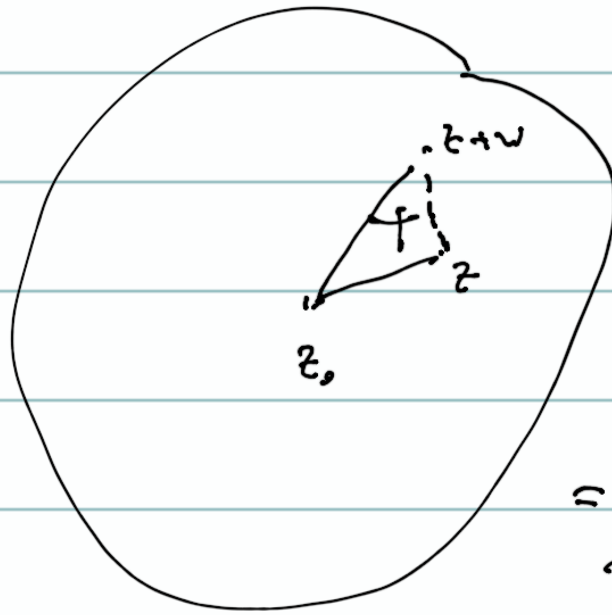
$$F(z) = \int_{[z_0, z]} f(\xi)\, d\xi$$

Proviamo che

$$F \in H(B_r(z_0))$$

e che $F'(z) = f(z)$

Sia $w \in \mathbb{C}$ t.c. $z+w \in B_r(z_0)$



$F(z+w) - F(z) =$

$$= \int_{[z_0, z+w]} - \int_{[z_0, z]} f(\xi)\, d\xi$$

$$= \underbrace{\int_{\partial T} f(\xi)\, d\xi}_{=0} - \underline{\int_{[z+w,\, z]} f(\xi)\, d\xi}$$

$$F(z+w) - F(z) = \int_{[z, z+w]} f(\xi)\, d\xi = \int_{[z, z+w]} \Big(f(z) + o(1)\Big)\, d\xi =$$

$w \to 0$.

$$= f(z) \int_{[z, z+w]} d\xi + o(|w|)$$

$$\int_{[z, z+w]} d\xi = \int_0^1 d(z_0 + tw) = \int_0^1 w\, dt = w$$

$$F(z+w) - F(z) = f(z)w + o(|w|)$$

$$\boxed{\frac{F(z+w) - F(z)}{w} = f(z) + o(1)} \qquad w \to 0.$$

Quindi F è diff. in senso complesso, ha per derivata f che è continua $\Rightarrow F \in H(B_r(z_0))$

$\Rightarrow f = F' \in H(B_r(z_0))$, per l'arb. di z_0

$f \in H(\Omega)$ $\square$