Conclusions

A striking consequence of the univocal exponential representability of any r.e. set was noted in [16, p. 300 and p. 310]. One can find a concrete polynomial $B(a, x_0, x_1, \ldots, x_{\boldsymbol{\kappa}}, y, w)$ with integral coefficients such that:

1) to each $\boldsymbol{a} \in \mathbb{N}$, there corresponds at most one $\boldsymbol{k}+2$ tuple $\langle \boldsymbol{v}_0, \boldsymbol{v}_1, \ldots, \boldsymbol{v}_{\boldsymbol{\kappa}}, \boldsymbol{u} \rangle$ such that $B(\boldsymbol{a}, \boldsymbol{v}_0, \boldsymbol{v}_1, \ldots, \boldsymbol{v}_{\boldsymbol{\kappa}}, \boldsymbol{u}, 2^{\boldsymbol{u}}) > 0$ holds;
2) to any monadic totally computable function $\mathcal{C}$, there correspond $\boldsymbol{k}+3$ tuples $\langle \boldsymbol{a}, \boldsymbol{v}_0, \boldsymbol{v}_1, \ldots, \boldsymbol{v}_{\boldsymbol{\kappa}}, \boldsymbol{u} \rangle$ of natural numbers such that
$$B(\boldsymbol{a}, \boldsymbol{v}_0, \boldsymbol{v}_1, \ldots, \boldsymbol{v}_{\boldsymbol{\kappa}}, \boldsymbol{u}, 2^{\boldsymbol{u}}) > 0 \text{ and } \max\{\boldsymbol{v}_0, \boldsymbol{v}_1, \ldots, \boldsymbol{v}_{\boldsymbol{\kappa}}, \boldsymbol{u}\} > \mathcal{C}(\boldsymbol{a}) .$$

To see this, refer to an explicit enumeration $\boldsymbol{f}_0, \boldsymbol{f}_1, \boldsymbol{f}_2, \ldots$ of all monadic partially computable functions (see [7, p. 73 ff]), so that both of

$$\mathcal{H} = \{ \langle a_1 , a_2 \rangle \in \mathbb{N}^2 \mid \boldsymbol{f}_{a_1}(a_1) = a_2 \} ,$$
$$\mathcal{K} = \{ a \in \mathbb{N} \mid \langle a , x \rangle \in \mathcal{H} \text{ holds for some } x \}$$

are r.e. sets, the complement $\mathbb{N} \setminus \mathcal{K}$ of the latter is not an r.e. set, and the former can be represented in the univocal form shown at the beginning of Sect. 1, namely

$$\boldsymbol{f}_{a_1}(a_1) = a_2 \iff (\exists x_1 \cdots \exists x_{\boldsymbol{\kappa}} \exists y \, \exists w)\big[2^y = w \;\&\; D(a_1, a_2 , x_1, \ldots, x_{\boldsymbol{\kappa}}, y, w) = 0\big],$$

where D is a polynomial with integral coefficients; then put

$$B(a, x_0, x_1, \ldots, x_{\boldsymbol{\kappa}}, y, w) =_{\text{Def}} 1 - D^2(a, x_0 , x_1, \ldots, x_{\boldsymbol{\kappa}}, y, w) ,$$

so that $B(a, x_0, x_1, \ldots, x_{\boldsymbol{\kappa}}, y, 2^y) > 0$ holds if and only if $\boldsymbol{f}_a(a) = x_0$, and hence B satisfies 1).

By way of contradiction, suppose that there is a monadic totally computable function $\mathcal{C}_*$ such that the inequalities $\boldsymbol{v_0} \leqslant \mathcal{C}_*(\boldsymbol{a}), \ldots, \boldsymbol{v}_{\boldsymbol{\kappa}} \leqslant \mathcal{C}_*(\boldsymbol{a})$, and $\boldsymbol{u} \leqslant \mathcal{C}_*(\boldsymbol{a})$ hold whenever a tuple $\langle \boldsymbol{a}, \boldsymbol{v}_0, \boldsymbol{v}_1, \ldots, \boldsymbol{v}_{\boldsymbol{\kappa}}, \boldsymbol{u} \rangle$ of natural numbers exists such that $B(\boldsymbol{a}, \boldsymbol{v}_0, \boldsymbol{v}_1, \ldots, \boldsymbol{v}_{\boldsymbol{\kappa}}, \boldsymbol{u}, 2^{\boldsymbol{u}}) > 0$ holds; that is, they hold when a pair $\langle \boldsymbol{a} , \boldsymbol{v}_0 \rangle \in \mathcal{H}$ exists (this happens, e.g., for the infinitely many $\boldsymbol{a}$'s satisfying $\mathcal{C}_* = \boldsymbol{f}_{\boldsymbol{a}}$). In particular, the said inequalities must hold when $\boldsymbol{a} \in \mathcal{K}$. But then this would offer us a criterion for checking whether or not $\boldsymbol{a} \in \mathcal{K}$, by evaluating a bounded family of expressions of the form $B(\boldsymbol{a}, v_0, v_1, \ldots, v_{\boldsymbol{\kappa}}, u, 2^u)$; however, this would conflict with the fact that $\mathbb{N} \setminus \mathcal{K}$ is not r.e. We conclude that B satisfies 2).

Summing up, we are in this situation: thanks to *reductio ad absurdum*, we have found that the course of values of the concrete arithmetic expression $B(a, v_0, v_1, \ldots, v_{\boldsymbol{\kappa}}, u, 2^u)$ exceeds zero at most once for each value $\boldsymbol{a}$ of a; it is unconceivable, though, that one can put an effective upper bound on the positive values of $B(a, v_0, v_1, \ldots, v_{\boldsymbol{\kappa}}, u, 2^u)$.

A proof that every r.e. set admits a finite-fold Diophantine polynomial representation would yield analogous, equally striking consequences about 'non-effectivizable estimates'.