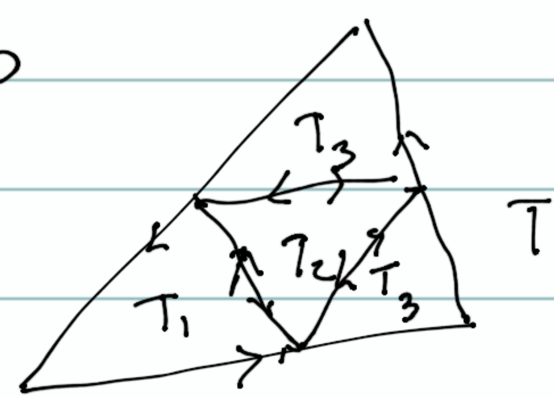
Def. $f : \Omega \subset \mathbb{C} \to \mathbb{C}$ si dice olomorfa se è diff. in senso complesso in ogni punto di Ω e f' è continua.

è sovrabbondante!

Teor di Goursat Se $f : \Omega \subset \mathbb{C} \to \mathbb{C}$ è diff. in ogni pto di Ω allora f è olomorfa.

Dim Sarà suff. dim. che per ogni triangolo T t.c. $\bar{T} \subset \Omega$ vale

$$\int_{\partial T} f(z)\,dz = 0$$



$$\partial T = \partial T_1 + \partial T_2 + \partial T_3 + \partial T_4$$

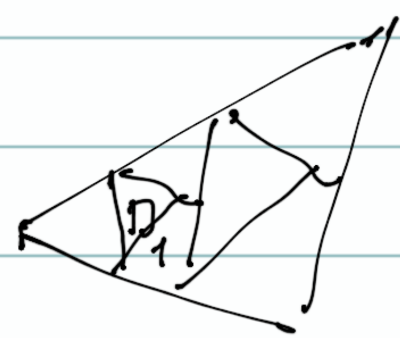
$$\int_{\partial T} f\,dz = \int_{\partial T_1} f\,dz + \dots + \int_{\partial T_4} f\,dz$$

$$\left|\int_{\partial T} ''\right| \leq \left|\int_{\partial T_1} ''\right| + \dots + \left|\int_{\partial T_4} ''\right|$$

Esiste $i = 1, 2, 3, 4$ t.c.

$$\triangle \quad \left|\int_{\partial T_i} f\,dz\right| \geq \frac{1}{4} \left|\int_{\partial T} f\,dz\right|$$

Scegliamo $D_0 = T$, D_1 il triangolino T_i che verifica $\triangle$



$D_0 = T$

Si suddivide D_1 in 4 triangoli e si ne seleziona uno (D_2) t.c.

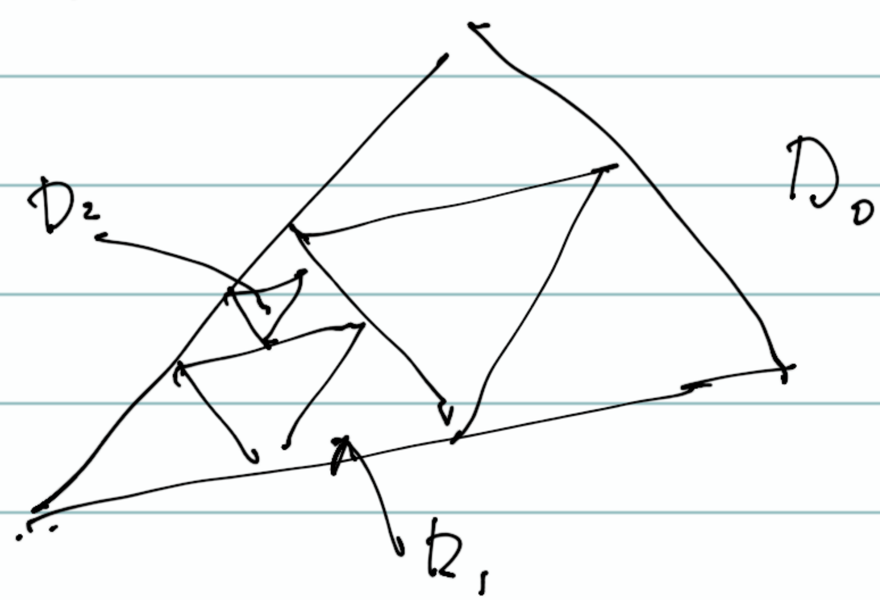
$$\left|\int_{\partial D_2} f(z)\,dz\right| \geq \frac{1}{4}\left|\int_{\partial D_1} f(z)\,dz\right|$$

$$\left|\int_{\partial D_1} f(z)\,dz\right| \geq \frac{1}{4}\left|\int_{\partial D_0} f(z)\,dz\right|$$

$$\left|\int_{\partial D_2} f(z)\,dz\right| \geq \frac{1}{4^2}\left|\int_{\partial D_0} f(z)\,dz\right|$$

$\operatorname{diam} D_1 = \frac{1}{2} \operatorname{diam} D_0$,

$\operatorname{diam} D_2 = \frac{1}{4} \operatorname{diam} D_0$.



Per induzione otteniamo una succ. di triangoli

D_n t.c. $\quad D_n \subset D_{n-1} \subset \dots \subset D_0$

$$\operatorname{diam} D_n = \frac{1}{2^n} \operatorname{diam} D_0$$

$$\left| \int_{\partial D_n} f(z)\,dz \right| \geq \frac{1}{4^n} \left| \int_{\partial D_0} f(z)\,dz \right|$$

Scegliamo per ogni D_n, un vertice, z_n

$$|z_n - z_{n-1}| \leq \frac{1}{2^{n-1}} \operatorname{diam} D_0 \quad , \quad \underline{\forall\, n, m \,,\, n \geq m!}$$

$$|z_n - z_m| \leq \left(\frac{1}{2^{n-1}} + \dots + \frac{1}{2^m} \right) \operatorname{diam} D_0$$

Quindi $z_n \to z_0 \in D_0$.

Inoltre per ogni $z \in \overline{D_n}$

$$z - z_0 \to 0 \quad \text{per } n \to \infty.$$

$$\int_{\partial D_n} f(z)\,dz = \underline{\int_{\partial D_n} \left(f(z) - f(z_0) \right) dz} + \underbrace{\int_{\partial D_n} dz}_{=0}\, f(z_0)$$

$$f(z)-f(z_0) = \underline{f'(z_0)(z-z_0)} + \underbrace{o(|z-z_0|)}$$

$z \in \partial D_n$

$$|z-z_0| \leq \operatorname{diam} D_n = \underline{2^{-n} \operatorname{diam} D_0}$$

$$\underline{\int_{\partial D_n} f(z)\,dz} = \boxed{\int_{\partial D_n} f'(z_0)(z-z_0)\,dz} = 0 + \int_{\partial D_n} o(|z-z_0|)\,dz$$

$$\left| \int_{\partial D_n} o(|z-z_0|)\,dz \right| \leq \underline{o(2^{-n} \operatorname{diam} D_0)}\ \underline{L(\partial D_n)} =$$

$$= o(2^{-n} \underline{\operatorname{diam} D_0})\, 2^{-n} \underline{L(\partial D_0)} =$$

$$= o(4^{-n})$$

$$\left| \int_{\partial D_0} f(z)\,dz \right| \leq 4^n \left| \int_{\partial D_n} f(z)\,dz \right| \leq 4^n o(4^{-n}) =$$

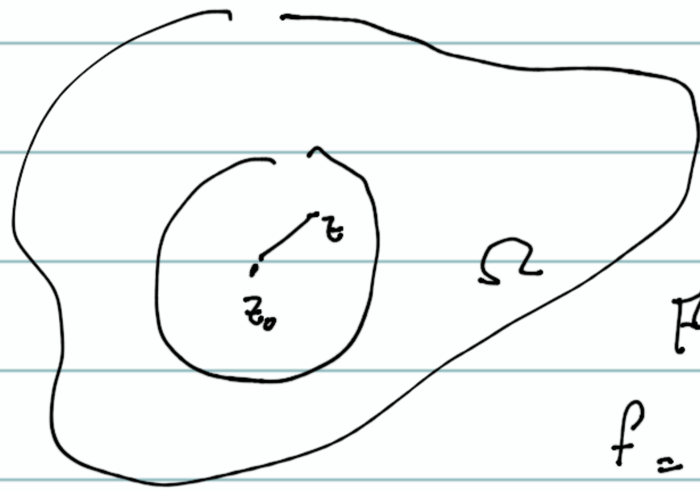
T

$$= o(1) \to 0 \qquad n \to \infty.$$

Quindi $\displaystyle\int_{\gamma_T} f(z)\,dz = 0 \quad \forall$ triangolo T

t.c. $T \subset \Omega$.

Per il Teor. di Morera $f \in H(\Omega)$. $\square$.



$$F(z) = \int_{[z_0, z]} f(\xi)\, d\xi$$

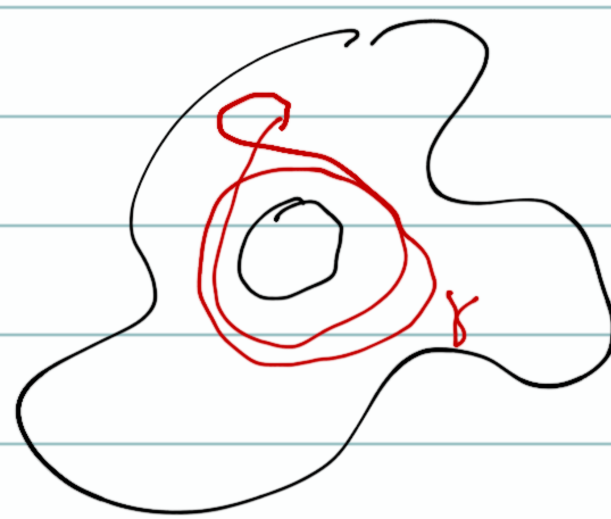
$F \in H(B_r(z_0))$.

$f = F'$

Oss. $f(z) = \frac{1}{z}$, $z \in \mathbb{C} \setminus \{0\}$. f non è dotata di primitive def. su tutto $\mathbb{C} \setminus \{0\}$.

Lemma. Sia $g \in H(\Omega)$, Ω aperto qual. di $\mathbb{C}$. Per ogni cammino chiuso γ t.c. $\gamma^* \subset \Omega$ vale

$$\int_\gamma g'(z)\, dz = 0$$



Dim. Sia $\gamma : [a,b] \to \mathbb{C}$ t.c.

$\gamma(a) = \gamma(b)$

$$\int_\gamma g'(z)\,dz = \int_a^b g'(\gamma(t))\,\gamma'(t)\,dt =$$

$$= \int_a^b \frac{d}{dt}\big(g(\gamma(t))\big)\,dt = 0$$

$$= g(\gamma(b)) - g(\gamma(a))$$

$\square$

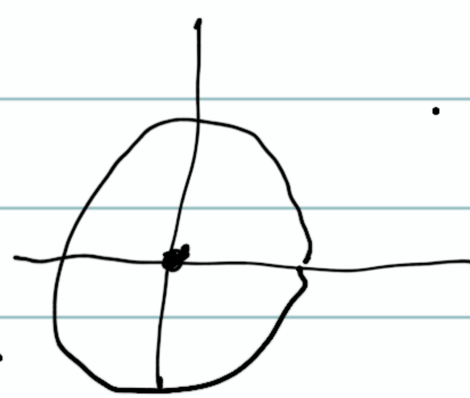
Tornando all'osservaz. di prima:

Se $f(z) = \frac{1}{z}$ avesse una primitiva F su $\mathbb{C} \setminus \{0\}$ si potrebbe prendere $\gamma =$ $= \partial B_r(0)$

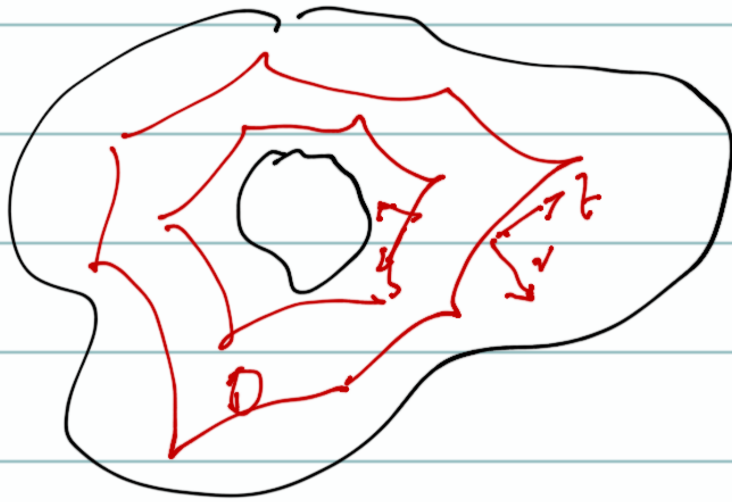
$$\int_{\partial B_r(0)} F'(z)\,dz = 0$$

$$\|$$

$$\int_{\partial B_r(0)} \frac{dz}{z} = 2\pi i \, \mathrm{Ind}_{\partial B_r(0)}(0) =$$



$'' = 2\pi i \neq 0$!!



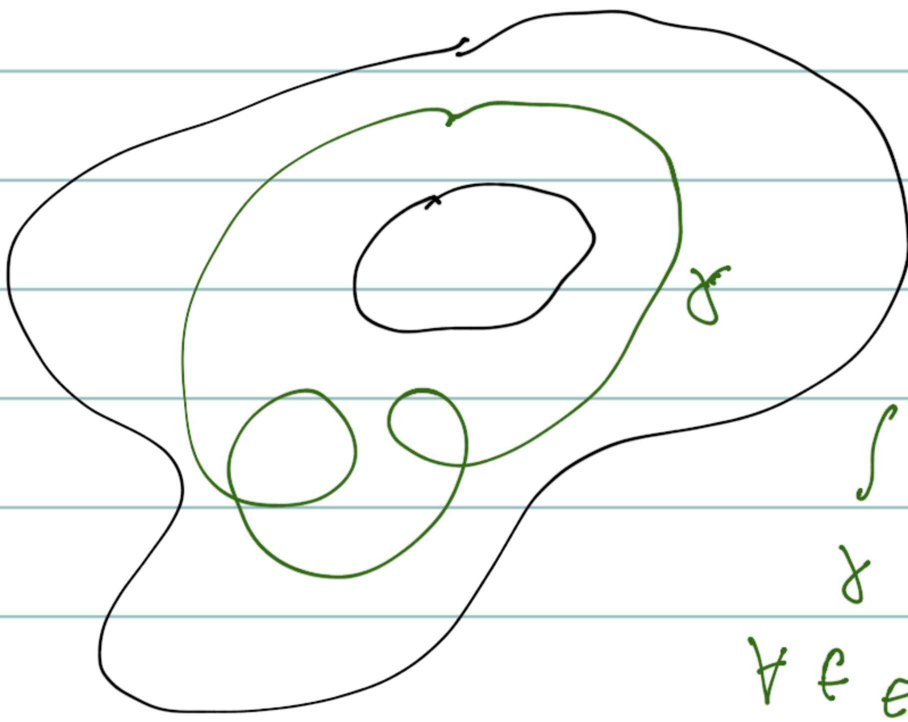
Ω

$\overline{D} \subset \Omega$

∂D reg. a tratti:

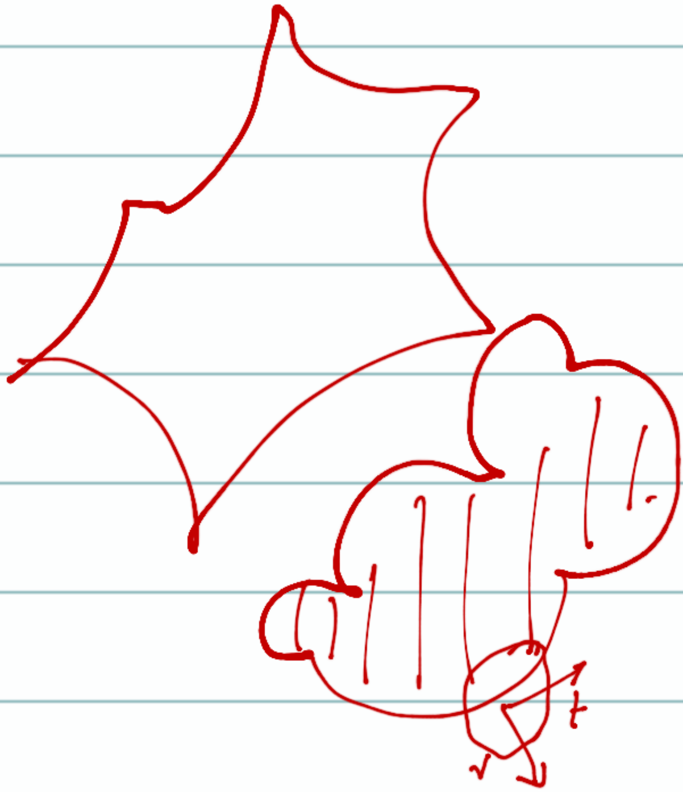
$$\int_{\partial D} f(z)\,dz = 0$$

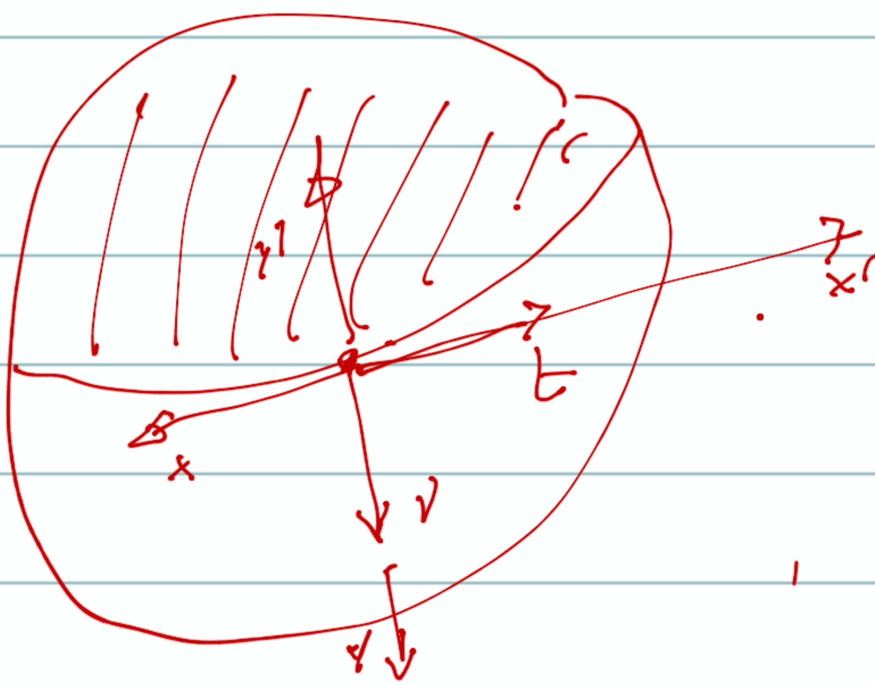
$\forall f \in H(\Omega)$



$$\int_{\gamma} f'(z)\,dz = 0$$

$\forall f \in H(\Omega)$


t
v


y'
x'
t
x
v
y

Teorema degli zeri Sia $D \subset \mathbb{C}$ un aperto connesso.
Sia $f \in H(D)$ non identicamente nulla.
Allora gli zeri di f sono punti isolati.

Lemma Nelle stesse ipotesi $\forall z_0 \in D$ $\exists$ $m = 0, 1, 2, \dots$
t.c. $f^{(m)}(z_0) \neq 0$.

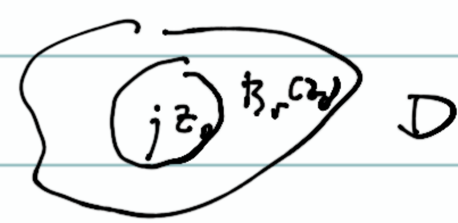
Dim del Lemma Proviamo

$$N = \{ z \in D \mid f^{(m)}(z) = 0 \quad \underline{\underline{\forall m}} \} =$$

$$= \bigcap_{m \geq 0}^{\infty} \underbrace{\{ z \in D \mid f^{(m)}(z) = 0 \}}_{\text{chiuso}}$$

N è chiuso $\subset D$. Proviamo che N è anche aperto.

Sia $z_0 \in N \subset D$



$\exists$ $r > 0$ t.c. $B_r(z_0) \subset D$. Quindi

$\forall z \in B_r(z_0)$

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n \equiv 0$$

quindi $f \equiv 0$ in $B_r(z_0) \Rightarrow f^{(m)}(z) = 0$

$\forall z \in B_r(z_0) \Rightarrow B_r(z_0) \subset N$!

cioè N è anche aperto. Dato che D è connesso, $N = \emptyset$ oppure $N = D$.

Se fosse $N = D$ avremmo $f \equiv 0$, impossibile. Quindi $N = \emptyset$ cioè $\forall z_0 \in D$ $\exists m$ t.c.

$$f^{(m)}(z_0) \neq 0. \quad \square$$

<u>Lemma</u> Nelle stesse ipotesi, $\forall z_0 \in D$ esiste $m = 0, 1, 2, \dots$ t.c. :

$$f(z) = \frac{f^{(m)}(z_0)}{m!}\underbrace{(z-z_0)^m} + O(|z-z_0|^{m+1})$$

con $f^{(m)}(z_0) \neq 0$.

<u>Dim.</u> Sia m il minimo indice t.c. $f^{(m)}(z_0) \neq 0$

$$f(z) = \boxed{\sum_{n=0}^{m-1} \frac{\underbrace{f^{(n)}(z_0)}_{=0}}{n!}(z-z_0)^n} + \frac{f^{(m)}(z_0)}{m!}(z-z_0)^m +$$

$$+ \underbrace{\sum_{n=m+1}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n} =$$

$$= \frac{f^{(m)}(z_0)}{m!}(z-z_0)^m +$$

$$+ (z-z_0)^{m+1}\overbrace{\left(\sum_{n=m+1}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z-z_0)^{n-m-1}\right)}^{\text{limitata vicino a } z_0}$$

z_0

$$|\ \|\ | \sim |(z-z_0)^{m+1}|$$

$$f(z) = \frac{f^{(m)}(z_0)}{m!}\underbrace{(z-z_0)^m}_{\neq 0} + O\left(|z-z_0|^{m+1}\right)$$

$\square$

Per $z \neq z_0$

$$\longrightarrow f(z) = \overbrace{\frac{f^{(m)}(z_0)}{m!}(z-z_0)^m}^{\neq 0}\Big(\overbrace{1 + O(|z-z_0|)}^{\neq 0}\Big)$$

Dim del Teorema: Se $f(z_0)=0$, allora nella form. vale $m \geq 1$

$$\Rightarrow f(z) = \frac{f^{(m)}(z_0)}{m!}(z-z_0)^m\left(1+O(|z-z_0|)\right)$$

quindi $\exists\, \rho>0$ t.c. $\forall\, z \neq z_0,\ |z-z_0|<\rho$

$f(z) \neq 0$. $\square$