7. Dimension.

Let X be a topological space.

7.1. Definition. The *topological dimension* of X is the supremum of the lengths of the chains of distinct irreducible closed subsets of X, where by definiton the following chain has length n:

$$X_0 \subset X_1 \subset X_2 \subset \ldots \subset X_n.$$

The topological dimension of X is denoted by $\dim X$. It is also called combinatorial or Krull dimension.

Example.
1. $\dim \mathbb{A}^1 = 1$: the maximal length chains have the form $\{P\} \subset \mathbb{A}^1$.
2. $\dim \mathbb{A}^n = n$: a chain of length n is

$$\{0\} = V(x_1, \ldots, x_n) \subset V(x_1, \ldots, x_{n-1}) \subset \ldots \subset V(x_1) \subset \mathbb{A}^n;$$

note that $V(x_1, \ldots, x_i)$ is irreducible for any $i \leq n$, because the ideal $\langle x_1, \ldots, x_i\rangle$ is prime. Indeed the quotient ring $K[x_1, \ldots, x_n]/\langle x_1, \ldots, x_i\rangle$ is isomorphic to $K[x_{i+1}, \ldots, x_n]$. Therefore $\dim \mathbb{A}^n \geq n$. On the other hand, from every chain of irreducible closed subsets of $\mathbb{A}^n$, passing to their ideals, we get a chain of the same length of prime ideals in $K[x_1, \ldots, x_n]$.

We define the Krull dimension of a ring A, and denote it by $\dim A$, to be the supremum of the lengths of the chains of distinct prime ideals of A. Therefore, we can reformulate the previous fact by saying that $\dim \mathbb{A}^n \leq \dim K[x_1, \ldots, x_n]$. We will see in a next chapter that $\dim K[x_1, \ldots, x_n] = n$. More in general, if A is a noetherian ring, then $\dim A[x] = \dim A + 1$.

3. Let X be irreducible. Then $\dim X = 0$ if and only if X is the closure of every point of it.

We prove now some useful relations between the dimension of X and the dimensions of its subspaces.

7.2. Proposition.
1. If $Y \subset X$, then $\dim Y \leq \dim X$. In particular, if $\dim X$ is finite, then also $\dim Y$ is (in this case, the number $\dim X - \dim Y$ is called the codimension of Y in X).

2. If $X = \bigcup_{i\in I} U_i$ is an open covering, then $\dim X = \sup\{\dim U_i\}$.

3. If X is noetherian and $X_1, \ldots, X_s$ are its irreducible components, then $\dim X = \sup_i \dim X_i$.

4. If $Y \subset X$ is closed, X is irreducible, $\dim X$ is finite and $\dim X = \dim Y$, then $Y = X$.

Proof.

1. Let $Y_0 \subset Y_1 \subset \ldots \subset Y_n$ be a chain of irreducible closed subsets of Y. Then their closures are irreducible and form the following chain: $\overline{Y_0} \subseteq \overline{Y_1} \subseteq \ldots \subseteq \overline{Y_n}$. Note that for all i $\overline{Y_i} \cap Y = Y_i$, because Y_i is closed into Y, so if $\overline{Y_i} = \overline{Y_{i+1}}$, then $Y_i = Y_{i+1}$. Therefore the two chains have the same length and we can conclude that $\dim Y \leq \dim X$.

2. Let $X_0 \subset X_1 \subset \ldots \subset X_n$ be a chain of irreducible closed subsets of X. Let $P \in X_0$ be a point: there exists an index $i \in I$ such that $P \in U_i$. So $\forall k = 0, \ldots, n$ $X_k \cap U_i \neq \emptyset$: it is an irreducible closed subset of U_i, irreducible because open in X_k which is irreducible. Consider $X_0 \cap U_i \subset X_1 \cap U_i \subset \ldots \subset X_n \cap U_i$; it is a chain of length n, because $\overline{X_k \cap U_i} = X_k$: in fact $X_k \cap U_i$ is open in X_k hence dense. Therefore, for all chain of irreducible closed subsets of X, there exists a chain of the same length of irreducible closed subsets of some U_i. So $\dim X \leq \sup \dim U_i$. By 1., equality holds.

3. Any chain of irreducible closed subsets of X is completely contained in an irreducible component of X. The conclusion follows as in 2.

4. If $Y_0 \subset Y_1 \subset \ldots \subset Y_n$ is a chain of maximal length in Y, then it is a maximal chain in X, because $\dim X = \dim Y$. Hence $X = Y_n \subset Y$. □

7.3. Corollary. $\dim \mathbb{P}^n = \dim \mathbb{A}^n$.

Proof. Because $\mathbb{P}^n = U_0 \cup \ldots \cup U_n$, and U_i is homeomorphic to $\mathbb{A}^n$ for all i. □

If X is noetherian and all its irreducible components have the same dimension r, then X is said to have *pure dimension* r.

Note that the topological dimension is invariant by homeomorphism. By definition, a *curve* is an algebraic set of pure dimension 1; a *surface* is an algebraic set of pure dimension 2.

We want to study the dimensions of affine algebraic sets. The following definition results to be very important.

7.4. Definition. Let $X \subset \mathbb{A}^n$ be an algebraic set. The *coordinate ring* of X is

$$K[X] := K[x_1, \ldots, x_n]/I(X).$$

It is a finitely generated K–algebra that has no non–zero nilpotents, because $I(X)$ is radical. This can be expressed by saying that $K[X]$ is a *reduced ring*. There is the canonical epimorphism $K[x_1, \ldots, x_n] \to K[X]$ such that $F \to [F]$. The elements of $K[X]$ can be interpreted as *polynomial functions* on X: to a polynomial F, we can associate the function $f : X \to K$ such that $P(a_1, \ldots, a_n) \to F(a_1, \ldots, a_n)$.

Two polynomials F, G define the same function on X if, and only if, $F(P) = G(P)$ for every point $P \in X$, i.e. if $F - G \in I(X)$, which means exactly that F and G have the same image in $K[X]$.

$K[X]$ is generated as K–algebra by $[x_1], \ldots, [x_n]$: these can be interpreted as the functions on X called *coordinate functions*, and generally denoted $t_1, \ldots, t_n$.

In fact $t_i : X \to K$ is the function which associates to $P(a_1, \ldots, a_n)$ the constant a_i. Note that the function f can be interpreted as $F(t_1, \ldots, t_n)$: the polynomial F evalued at the n– tuple of the coordinate functions.

In the projective space we can do an analogous construction. If $Y \subset \mathbb{P}^n$ is closed, then the *homogeneous coordinate ring* of Y is

$$S(Y) := K[x_0, x_1, \ldots, x_n]/I_h(Y).$$

It is also a finitely generated K–algebra, but its elements have no interpretation as functions on Y. They are functions on the cone $C(Y)$.

7.5. Definition. Let R be a ring. The *Krull dimension* of R is the supremum of the lengths of the chains of prime ideals of R

$$\mathcal{P}_0 \subset \mathcal{P}_1 \subset \ldots \subset \mathcal{P}_r.$$

Similarly, the *heigth* of a prime ideal $\mathcal{P}$ is the sup of the lengths of the chains of prime ideals contained in $\mathcal{P}$: it is denoted ht$\mathcal{P}$.

7.6. Proposition. *Let K be an algebraically closed field. Let X be an affine algebraic set contained in $\mathbb{A}^n$. Then* $\dim X = \dim K[X]$.

Proof.
By the Nullstellensatz and by 6.5 the chains of irreducible closed subsets of X correspond bijectively to the chains of prime ideals of $K[x_1, \ldots, x_n]$ containing $I(X)$, hence to the chains of prime ideals of the quotient ring $K[X]$. □

The dimension theory for commutative rings contains some important theorems about dimension of K–algebras. The following two results are very useful.

7.7. Theorem. *Let K be any field.*

1. Let B be a finitely generated K–algebra and an integral domain. Then $\dim B = tr.d.Q(B)/K$*, where $Q(B)$ is the quotient field of B. In particular* $\dim B$ *is finite.*

2. Let B be as above and $\mathcal{P} \subset B$ be any prime ideal. Then $\dim B = \mathrm{ht}\mathcal{P} + \dim B/\mathcal{P}$.

Proof. For 1. see Portelli's notes. For a proof of 2., see for instance [4], Ch. II, Proposition 3.4. It relies on the normalization lemma and the lying over theorem. □

7.8. Corollary. *Let K be an algebraically closed field.*

1. $\dim \mathbb{A}^n = \dim \mathbb{P}^n = n$.

2. If X is an affine variety, then $\dim X = tr.d.K(X)/K$*, where $K(X)$ denotes the quotient field of $K[X]$.*

2. If $X \subset \mathbb{A}^n$ *is closed and irreducible, then* $\dim X = n - \mathrm{ht} I(X)$. □

The following is an important characterization of the algebraic subsets of $\mathbb{A}^n$ of codimension 1.

7.9. Proposition. *Let* $X \subset \mathbb{A}^n$ *be closed. Then* X *is a hypersurface if and only if* X *is of pure dimension* $n - 1$.

Proof. We give here an elementary direct proof. It can be proved more quickly using the Krull principal ideal theorem.

Let $X \subset \mathbb{A}^n$ be a hypersurface, with $I(X) = (F) = (F_1 \dots F_s)$, where $F_1, \dots, F_s$ are the irreducible factors of F all of multiplicity one. Then $V(F_1), \dots, V(F_s)$ are the irreducible components of X, whose ideals are (F_1), ..., (F_s). So it is enough to prove that $\mathrm{ht}(F_i) = 1$, for $i = 1, \dots, s$.

If $\mathcal{P} \subset (F_i)$ is a prime ideal, then either $\mathcal{P} = (0)$ or there exists $G \in \mathcal{P}$, $G \neq 0$. In the second case, let A be an irreducible factor of G belonging to $\mathcal{P}$: $A \in (F_i)$ so $A = HF_i$. Since A is irreducible, either H or F_i is invertible; F_i is irreducible, so H is invertible, hence $(A) = (F_i) \subset \mathcal{P}$. Therefore either $\mathcal{P} = (0)$ or $\mathcal{P} = (F_i)$, and $\mathrm{ht}(F_i) = 1$.

Conversely, assume that X is irreducible of dimension $n - 1$. Since $X \neq \mathbb{A}^n$, there exists $F = F_1 \dots F_s \in I(X)$, $F \neq 0$. Hence $X \subset V(F) = V(F_1) \cup \dots \cup V(F_s)$. By the irreducibility of X, some irreducible factor of F, call it F_i, also vanishes along X. Therefore $X \subset V(F_i)$, which is irreducible of dimension $n - 1$, by the first part. So $X = V(F_i)$ (by Proposition 7.2, 3). □

This proposition does not generalize to higher codimension. There exist codimension 2 algebraic subsets of $\mathbb{A}^n$ whose ideal is not generated by two polynomials. An example in $\mathbb{A}^3$ is the curve X parametrized by (t^3, t^4, t^5). A system of generators of $I(X)$ is $\langle x^3 - yz, y^2 - xz, z^2 - x^2y \rangle$. One can easily show that $I(X)$ cannot be generated by two polynomials. For a discussion of this and other similar examples, see [4], Chapter V.

7.10. Proposition. *Let* $X \subset \mathbb{A}^n$, $Y \subset \mathbb{A}^m$ *be irreducible closed subsets. Then* $\dim X \times Y = \dim X + \dim Y$.

Proof. Let $r = \dim X$, $s = \dim Y$; let $t_1, \dots, t_n$ (resp. $u_1, \dots, u_m$) be coordinate functions on $\mathbb{A}^n$ (resp. $\mathbb{A}^m$). We can assume that $t_1, \dots, t_r$ be a transcendence basis of $Q(K[X])$ and $u_1, \dots, u_s$ be a transcendence basis of $Q(K[Y])$. By definition, $K[X \times Y]$ is generated as K–algebra by $t_1, \dots, t_n, u_1, \dots, u_m$: we want to show that $t_1, \dots, t_r, u_1, \dots, u_s$ is a transcendence basis of $Q(K[X \times Y])$ over K. Assume that $F(x_1, \dots, x_r, y_1, \dots, y_s)$ is a polynomial which vanishes on $t_1, \dots, t_r, u_1, \dots, u_s$, i.e. F defines the zero function on $X \times Y$. Then, $\forall\ P \in X$, $F(P; y_1, \dots, y_s)$ is zero on Y, i.e. $F(P; u_1, \dots, u_s) = 0$. Since $u_1, \dots, u_s$ are algebraically independent, every coefficient $a_i(P)$ of $F(P; y_1, \dots, y_s)$ is zero, $\forall\ P \in X$. Since $t_1, \dots, t_r$ are algebraically independent, the polynomials $a_i(x_1, \dots, x_r)$ are zero, so

$F(x_1, \dots, x_r, y_1, \dots, y_s) = 0$. So $t_1, \dots, t_r, u_1, \dots, u_s$ are algebraically independent. Since this is certainly a maximal algebraically free set, it is a transcendence basis. □

Exercises to §7.

1*. Prove that a proper closed subset of an irreducible curve is a finite set. Deduce that any bijection between irreducible curves is a homeomorphism.

2*. Let $X \subset \mathbb{A}^2$ be the cuspidal cubic of equation: $x^3 - y^2 = 0$, let $K[X]$ be its coordinate ring. Prove that all elements of $K[X]$ can be written in a unique way in the form $f(x) + yg(x)$, where f, g are polynomial in the variable x. Deduce that $K[X]$ is not isomorphic to a polynomial ring.

8. Regular and rational functions.

a) Regular functions

Let $X \subset \mathbb{P}^n$ be a locally closed subset and P be a point of X. Let $\phi : X \to K$ be a function.

8.1. Definition. ϕ is *regular at* P if there exists a suitable neighbourhood of P in which ϕ can be expressed as a quotient of homogeneous polynomials of the same degree; more precisely, if there exist an open neighbourhood U of P in X and homogeneous polynomials F, $G \in K[x_0, x_1, \dots, x_n]$ with $\deg F = \deg G$, such that $U \cap V_P(G) = \emptyset$ and $\phi(Q) = F(Q)/G(Q)$, for all $Q \in U$. Note that the quotient $F(Q)/G(Q)$ is well defined.

ϕ is *regular on* X if ϕ is regular at every point P of X.

The set of regular functions on X is denoted $\mathcal{O}(X)$: it contains K (identified with the set of constant functions), and can be given the structure of a K–algebra, by the definitions:

$$(\phi + \psi)(P) = \phi(P) + \psi(P)$$

$$(\phi\psi)(P) = \phi(P)\psi(P),$$

for $P \in X$. (Check that $\phi + \psi$ and $\phi\psi$ are indeed regular on X.)

8.2. Proposition. *Let $\phi : X \to K$ be a regular function. Let K be identified with $\mathbb{A}^1$ with Zariski topology. Then ϕ is continuous.*

Proof. It is enough to prove that $\phi^{-1}(c)$ is closed in X, $\forall\, c \in K$. For all $P \in X$, choose an open neighbourhood U_P and homogeneous polynomials F_P, G_P such that $\phi|_P = F_P/G_P$. Then

$$\phi^{-1}(c) \cap U_P = \{Q \in U_P | F_P(Q) - cG_P(Q) = 0\} = U_P \cap V_P(F_P - cG_P)$$

is closed in U_P. The proposition then follows from:

8.3. Lemma. *Let T be a topological space, $T = \cup_{i\in I} U_i$ be an open covering of T, $Z \subset T$ be a subset. Then Z is closed if and only if $Z \cap U_i$ is closed in U_i for all i.*

Proof. Assume that $U_i = X \setminus C_i$ and $Z \cap U_i = Z_i \cap U_i$, with C_i and Z_i closed in X.

Claim: $Z = \bigcap_{i\in I}(Z_i \cup C_i)$, hence it is closed.
In fact: if $P \in Z$, then $P \in Z \cap U_i$ for a suitable i. Therefore $P \in Z_i \cap U_i$, so $P \in Z_i \cup C_i$. If $P \notin Z_j \cap U_j$ for some j, then $P \notin U_j$ so $P \in C_j$ and therefore $P \in Z_j \cup C_j$.

Conversely, if $P \in \bigcap_{i\in I}(Z_i \cup C_i)$, then $\forall\, i$, either $P \in Z_i$ or $P \in C_i$. Since $\exists j$ such that $P \in U_j$, hence $P \notin C_j$, so $P \in Z_j$, so $P \in Z_j \cap U_j = Z \cap U_j$.

□

8.4. Corollary.

1. Let $\phi \in \mathcal{O}(X)$: then $\phi^{-1}(0)$ is closed. It is denoted $V(\phi)$ and called the set of zeroes of ϕ.

2. Let X be a quasi–projective variety and ϕ, $\psi \in \mathcal{O}(X)$. Assume that there exists U, open non –empty subset such that $\phi|_U = \psi|_U$. Then $\phi = \psi$.

Proof. $\phi - \psi \in \mathcal{O}(X)$ so $V(\phi - \psi)$ is closed. By assumption $V(\phi - \psi) \supset U$, which is dense, because X is irreducible. So $V(\phi - \psi) = X$.

□

If $X \subset \mathbb{A}^n$ is locally closed, we can use on X both homogeneous and non–homogeneous coordinates. In the second case, a regular function is locally represented as a quotient F/G, with F and $G \in K[x_1, \dots, x_n]$. In particular all polynomial functions are regular, so, if X is closed, $K[X] \subset \mathcal{O}(X)$.

If $\alpha \subset K[X]$ is an ideal, we can consider $V(\alpha) := \bigcap_{\phi\in\alpha} V(\phi)$: it is closed into X. Note that α is of the form $\alpha = \overline{\alpha}/I(X)$, where $\overline{\alpha}$ is the inverse image of α in the canonical epimorphism, it is an ideal of $K[x_1, \dots, x_n]$ containing $I(X)$, hence $V(\alpha) = V(\overline{\alpha}) \cap X = V(\overline{\alpha})$.

If K is algebraically closed, from the Nullstellensatz it follows that, if α is proper, then $V(\alpha) \neq \emptyset$. Moreover the following relative form of the Nullstellensatz holds: if $f \in K[X]$ and f vanishes at all points $P \in X$ such that $g_1(P) = \dots = g_m(P) = 0$ ($g_1, \dots, g_m \in K[X]$), then $f^r \in \langle g_1, \dots, g_m \rangle \subset K[X]$, for some $r \geq 1$.

8.5. Theorem. *Let K be an algebraically closed field. Let $X \subset \mathbb{A}^n_K$ be closed in the Zariski topology. Then $\mathcal{O}(X) \simeq K[X]$. It is an integral domain if and only if X is irreducible.*

Proof. Let $f \in \mathcal{O}(X)$.

(i) Assume first that X is irreducible. For all $P \in X$ fix an open neighbourhood U_P of P and polynomials F_P, G_P such that $V_P(G_P) \cap U_P = \emptyset$ and $f|_{U_P} = F_P/G_P$. Let f_P, g_P be the functions in $K[X]$ defined by F_P and G_P. Then $g_P f = f_P$ holds on U_P, so it holds on X (by Corollary 8.3, because X is irreducible). Let $\alpha \subset K[X]$ be the ideal $\alpha = \langle g_P \rangle_{P \in X}$; α has no zeroes on X, because $g_P(P) \neq 0$, so $\alpha = K[X]$. Therefore there exists $h_P \in K[X]$ such that $1 = \sum_{P \in X} h_P g_P$ (sum with finite support). Hence in $\mathcal{O}(X)$ we have the relation: $f = f \sum h_P g_P = \sum h_P (g_P f) = \sum h_P f_P \in K[X]$.

(ii) Let X be reducible: for any $P \in X$, there exists $R \in K[x_1, \ldots, x_n]$ such that $R(P) \neq 0$ and $R \in I(X \setminus U_P)$, so $r \in \mathcal{O}(X)$ is zero outside U_P. So $r g_P f = f_P r$ on X and we conclude as above by replacing g_P with $g_P r$ and f_P with $f_P r$.

□

The characterization of regular functions on projective varieties is completely different: we will see in §12 that, if X is a projective variety, then $\mathcal{O}(X) \simeq K$, i.e. the unique regular functions are constant.

This gives the motivation for introducing the following weaker concept.

b) Rational functions

8.6. Definition. Let X be a quasi–projective variety. A *rational function* on X is a germ of regular functions on some open non–empty subset of X.

Precisely, let $\mathcal{K}$ be the set $\{(U, f) | U \neq \emptyset, \text{open subset of } X, f \in \mathcal{O}(U)\}$. The following relation on $\mathcal{K}$ is an equivalence relation:

$$(U, f) \sim (U', f') \text{ if and only if } f|_{U \cap U'} = f'|_{U \cap U'}.$$

Reflexive and symmetric properties are quite obvious. Transitive property: let $(U, f) \sim (U', f')$ and $(U', f') \sim (U'', f'')$. Then $f|_{U \cap U'} = f'|_{U \cap U'}$ and $f'|_{U' \cap U''} = f''|_{U' \cap U''}$, hence $f|_{U \cap U' \cap U''} = f''|_{U \cap U' \cap U''}$. $U \cap U' \cap U''$ is a non–empty open subset of $U \cap U''$ (which is irreducible and quasi–projective), so by Corollary 8.4 $f|_{U' \cap U''} = f''|_{U' \cap U''}$.

Let $K(X) := \mathcal{K}/\sim$: its elements are by definition rational functions on X. $K(X)$ can be given the structure of a field in the following natural way.

Let $\langle U, f \rangle$ denote the class of (U, f) in $K(X)$. We define:

$$\langle U, f \rangle + \langle U', f' \rangle = \langle U \cap U', f + f' \rangle,$$

$$\langle U, f \rangle \langle U', f' \rangle = \langle U \cap U', f f' \rangle$$

(check that the definitions are well posed!).

There is a natural inclusion: $K \to K(X)$ such that $c \to \langle X, c \rangle$. Moreover, if $\langle U, f \rangle \neq 0$, then there exists $\langle U, f \rangle^{-1} = \langle U \setminus V(f), f^{-1} \rangle$: the axioms of a field are all satisfied.

There is also an injective map: $\mathcal{O}(X) \to K(X)$ such that $\phi \to \langle X, \phi\rangle$.

8.7. Proposition. *If $X \subset \mathbb{A}^n$ is affine, then $K(X) \simeq Q(\mathcal{O}(X)) = K(t_1, \dots, t_n)$, where $t_1, \dots, t_n$ are the coordinate functions on X.*

Proof. The isomorphism is as follows:

(i) $\psi : K(X) \to Q(\mathcal{O}(X))$

If $\langle U, \phi\rangle \in K(X)$, then there exists $V \subset U$, open and non–empty, such that $\phi\ |_V = F/G$, where $F, G \in K[x_1, \dots, x_n]$ and $V(G) \cap V = \emptyset$. We set $\psi(\langle U, \phi\rangle) = f/g$.

(ii) $\psi' : Q(\mathcal{O}(X)) \to K(X)$

If $f/g \in Q(\mathcal{O}(X))$, we set $\psi'(f/g) = \langle X \setminus V(g), f/g\rangle$.

It is easy to check that ψ and ψ' are well defined and inverse each other. □

8.8. Corollary. *If X is an affine variety, then $\dim X$ is equal to the transcendence degree over K of its field of rational functions..*

8.9. Proposition. *If X is quasi–projective and $U \neq \emptyset$ is an open subset, then $K(X) \simeq K(U)$.*

Proof. We have the maps: $K(U) \to K(X)$ such that $\langle V, \phi\rangle \to \langle V, \phi\rangle$, and $K(X) \to K(U)$ such that $\langle A, \psi\rangle \to \langle A \cap U, \psi\ |_{A\cap U}\rangle$: they are K–homomorphisms inverse each other. □

8.10. Corollary. *If X is a projective variety contained in $\mathbb{P}^n$, if i is an index such that $X \cap U_i \neq \emptyset$ (where U_i is the open subset where $x_i \neq 0$), then $\dim X = \dim X \cap U_i = tr.d.K(X)/K$.*

Proof. By Proposition 7.2 $\dim X = \sup \dim(X \cap U_i)$. By 8.8 and 8.9, if $X \cap U_i$ is non–empty, $\dim(X \cap U_i) = tr.d.K(X \cap U_i)/K = tr.d.K(X)/K$ is independent of i. □

If $\langle U, \phi\rangle \in K(X)$, we can consider all possible representatives of it, i.e. all pairs $\langle U_i, \phi_i\rangle$ such that $\langle U, \phi\rangle = \langle U_i, \phi_i\rangle$. Then $\overline{U} = \bigcup_i U_i$ is the maximum open subset of X on which ϕ can be seen as a function: it is called the *domain of definition* (or of regularity) of $\langle U, \phi\rangle$, or simply of ϕ. It is sometimes denoted domϕ. If $P \in \overline{U}$, we say that *ϕ is regular at P.*

We can consider the set of rational functions on X which are regular at P: it is denoted by $\mathcal{O}_{P,X}$. It is a subring of $K(X)$ containing $\mathcal{O}(X)$, called the *local ring of X at P*. In fact, $\mathcal{O}_{P,X}$ is a local ring, whose maximal ideal, denoted $\mathcal{M}_{P,X}$, is the set of rational functions ϕ such that $\phi(P)$ is defined and $\phi(P) = 0$. To see this, observe that an element of $\mathcal{O}_{P,X}$ can be represented as $\langle U, F/G\rangle$: its inverse in $K(X)$ is $\langle U \setminus V_P(G), G/F\rangle$, which belongs to $\mathcal{O}_{P,X}$ if and only if $F(P) \neq 0$. We'll see in 8.12 that $\mathcal{O}_{P,X}$ is the localization $K[X]_{I_X(P)}$.

As in Proposition 8.9 for the fields of rational functions, also for the local rings of points it can easily be proved that, if $U \neq \emptyset$ is an open subset of X containing P, then $\mathcal{O}_{P,X} \simeq \mathcal{O}_{P,U}$. So the ring $\mathcal{O}_{P,X}$ only depends on the local behaviour of X in the neighbourhood of P.

The *residue field* of $\mathcal{O}_{P,X}$ is the quotient $\mathcal{O}_{P,X}/\mathcal{M}_{P,X}$: it is a field which results to be naturally isomorphic to the base field K. In fact consider the evaluation map $\mathcal{O}_{P,X} \to K$ such that ϕ goes to $\phi(P)$: it is surjective with kernel $\mathcal{M}_{P,X}$, so $\mathcal{O}_{P,X}/\mathcal{M}_{P,X} \simeq K$.

8.11. Examples.

1. Let $Y \subset \mathbb{A}^2$ be the curve $V(x_1^3 - x_2^2)$. Then $F = x_2$, $G = x_1$ define the function $\phi = x_2/x_1$ which is regular at the points $P(a_1, a_2)$ such that $a_1 \neq 0$. Another representation of the same function is: $\phi = x_1^2/x_2$, which shows that ϕ is regular at P if $a_2 \neq 0$. If ϕ admits another representation F'/G', then $G'x_2 - F'x_1$ vanishes on an open subset of X, which is irreducible (see Exercise 6.2), hence $G'x_2 - F'x_1$ vanishes on X, and therefore $G'x_2 - F'x_1 \in \langle x_1^3 - x_2^2 \rangle$. This shows that there are essentially only the above two representations of ϕ. So $\phi \in K(X)$ and its domain of regularity is $Y \setminus \{0, 0\}$.

2. The stereographic projection.
Let $X \subset \mathbb{P}^2$ be the curve $V_P(x_1^2+x_2^2-x_0^2)$. Let $f := x_1/(x_0-x_2)$ denote the germ of the regular function defined by $x_1/(x_0-x_2)$ on $X \setminus V_P(x_0-x_2) = X \setminus \{[1,0,1]\} = X \setminus \{P\}$. On X we have $x_1^2 = (x_0 - x_2)(x_0 + x_2)$ so f is represented also as $(x_0 + x_2)/x_1$ on $X \setminus V_P(x_1) = X \setminus \{P, Q\}$, where $Q = [1, 0, -1]$. If we identify K with the affine line $V_P(x_2) \setminus V_P(x_0)$ (the points of the x_1–axis lying in the affine plane U_0), then f can be interpreted as the stereographic projection of X centered at P, which takes $A[a_0, a_1, a_2]$ to the intersection of the line AP with the line $V_P(x_2)$. To see this, observe that AP has equation $a_1x_0 + (a_2 - a_0)x_1 - a_1x_2 = 0$; and $AP \cap V_P(x_2)$ is the point $[a_0 - a_2, a_1, 0]$.

8.12. The algebraic characterization of the local ring $\mathcal{O}_{P,X}$.

Let us recall the construction of the *ring of fractions of a ring* A with respect to a multiplicative subset S.

Let A be a ring and $S \subset A$ be a multiplicative subset. The following relation in $A \times S$ is an equivalence relation:

$$(a, s) \simeq (b, t) \text{ if and only if } \exists u \in S \text{ such that } u(at - bs) = 0.$$

Then the quotient $A \times S/_{\simeq}$ is denoted $S^{-1}A$ or A_S and $[(a, s)]$ is denoted $\frac{a}{s}$. A_S becomes a commutative ring with unit with operations $\frac{a}{s} + \frac{b}{t} = \frac{at+bs}{st}$ and $\frac{a}{s}\frac{b}{t} = \frac{ab}{st}$ (check that they are well–defined). With these operations, A_S is called the ring of fractions of A with respect to S, or the *localization* of A in S.

There is a natural homomorphism $j : A \to S^{-1}A$ such that $j(a) = \frac{a}{1}$, which makes $S^{-1}A$ an A–algebra. Note that j is the zero map if and only if $0 \in S$. More precisely if $0 \in S$ then $S^{-1}A$ is the zero ring: this case will always be excluded in what follows. Moreover j is injective if and only if every element in S is not a zero divisor. In this case $j(A)$ will be identified with A.

Examples.

1. Let A be an integral domain and set $S = A \setminus \{0\}$. Then $A_S = Q(A)$: the quotient field of A.

2. If $\mathcal{P} \subset A$ is a prime ideal, then $S = A \setminus \mathcal{P}$ is a multiplicative set and A_S is denoted $A_{\mathcal{P}}$ and called the localization of A at $\mathcal{P}$.

3. If $f \in A$, then the multiplicative set generated by f is

$$S = \{1, f, f^2, \dots, f^n, \dots\} :$$

A_S is denoted A_f.

4. If $S = \{x \in A \mid x \text{ is regular}\}$, then A_S is called the total ring of fractions of A: it is the maximum ring in which A can be canonically embedded.

It is easy to verify that the ring A_S enjoys the following *universal property*:

(i) if $s \in S$, then $j(s)$ is invertible;

(ii) if B is a ring with a given homomorphism $f : A \to B$ such that if $s \in S$, then $f(s)$ is invertible, then f factorizes through A_S, i.e. there exists a unique homomorphism $\overline{f}$ such that $\overline{f} \circ j = f$.

We will see now the relations between ideals of A_S and ideals of A.

If $\alpha \subset A$ is an ideal, then $\alpha A_S = \{\frac{a}{s} \mid a \in \alpha\}$ is called the *extension of* α in A_S and denoted also α^e. It is an ideal, precisely the ideal generated by the set $\{\frac{a}{1} \mid a \in \alpha\}$.

If $\beta \subset A_S$ is an ideal, then $j^{-1}(\beta) =: \beta^c$ is called the contraction of β and is clearly an ideal.

We have:

8.13. Proposition.

1. $\forall \alpha \subset A : \alpha^{ec} \supset \alpha$*;*

2. $\forall \beta \subset A_S : \beta = \beta^{ce}$*;*

3. α^e *is proper if and only if* $\alpha \cap S = \emptyset$*;*

4. $\alpha^{ec} = \{x \in A \mid \exists s \in S \text{ such that } sx \in \alpha\}$.

Proof.

1. and 2. are straightforward.

3. if $1 = \frac{a}{s} \in \alpha^e$, then there exists $u \in S$ such that $u(s - a) = 0$, i.e. $us = ua \in S \cap \alpha$. Conversely, if $s \in S \cap \alpha$ then $1 = \frac{s}{s} \in \alpha^e$.

4.

$$\alpha^{ec} = \{x \in A \mid j(x) = \frac{x}{1} \in \alpha^e\} =$$

$$= \{x \in A \mid \exists a \in \alpha, t \in S \text{ such that } \frac{x}{1} = \frac{a}{t}\} =$$

$$= \{x \in A \mid \exists a \in \alpha, t, u \in S \text{ such that } u(xt - a) = 0\}.$$

Hence, if $x \in \alpha^{ec}$, then: $(ut)x = ua \in \alpha$. Conversely: if there exists $s \in S$ such that $sx = a \in \alpha$, then $\frac{x}{1} = \frac{a}{s}$, i.e. $j(x) \in \alpha^e$. □

If α is an ideal of A such that $\alpha = \alpha^{ec}$, α is called *saturated* with S. For example, if $\mathcal{P}$ is a prime ideal and $S \cap \mathcal{P} = \emptyset$, then $\mathcal{P}$ is saturated and $\mathcal{P}^e$ is prime. Conversely, if $\mathcal{Q} \subset A_S$ is a prime ideal, then $\mathcal{Q}^c$ is prime in A.

Therefore: *there is a bijection between the set of prime ideals of A_S and the set of prime ideals of A not intersecting S. In particular, if $S = A \setminus \mathcal{P}$, $\mathcal{P}$ prime, the prime ideals of $A_\mathcal{P}$ correspond bijectively to the prime ideals of A contained in $\mathcal{P}$, hence $A_\mathcal{P}$ is a local ring with maximal ideal $\mathcal{P}^e$, denoted $\mathcal{P}A_\mathcal{P}$, and residue field $A_\mathcal{P}/\mathcal{P}A_\mathcal{P}$.* Moreover $\dim A_\mathcal{P} = \text{ht}\mathcal{P}$.

In particular we get the characterization of $\mathcal{O}_{P,X}$. Let $X \subset \mathbb{A}^n$ be an affine variety, let P be a point of X and $I(P) \subset K[x_1, \ldots, x_n]$ be the ideal of P. Let $I_X(P) := I(P)/I(X)$ be the ideal of $K[X]$ formed by regular functions on X vanishing at P. Then we can construct the localization

$$\mathcal{O}(X)_{I_X(P)} = \{\frac{f}{g} | f, g \in \mathcal{O}(X), g(P) \neq 0\} \subset K(X) :$$

it is canonically identified with $\mathcal{O}_{P,X}$. In particular: $\dim \mathcal{O}_{P,X} = \text{ht } I_X(P) = \dim \mathcal{O}(X) = \dim X$.

There is a bijection between prime ideals of $\mathcal{O}_{P,X}$ and prime ideals of $\mathcal{O}(X)$ contained in $I_X(P)$; they also correspond to prime ideals of $K[x_1, \ldots, x_n]$ contained in $I(P)$ and containing $I(X)$.

If X is affine, it is possible to define the local ring $\mathcal{O}_{P,X}$ also if X is reducible, simply as localization of $K[X]$ at the maximal ideal $I_X(P)$. The natural map j from $K[X]$ to $\mathcal{O}_{P,X}$ is injective if and only if $K[X] \setminus I_X(P)$ does not contain any zero divisor. A non-zero function f is a zero divisor in $K[X]$ if there exists a non-zero g such that $fg = 0$, i.e. $X = V(f) \cup V(g)$ is an expression of X as union of proper closed subsets. For j to be injective it is required that every zero divisor f belongs to $I_X(P)$, which means that all the irreducible components of X pass through P.

Exercises to §8.

1. Prove that the affine varieties and the open subsets of affine varieties are quasi–projective.

2. Let $X = \{P, Q\}$ be the union of two points in an affine space over K. Prove that $\mathcal{O}(X)$ is isomorphic to $K \times K$.