Lecture 1-2
Introduction to decision theory

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Preferences and Utility function

Example:

Consider the following items:

Apple, Milk, Chocolate, Ice cream

What do you prefer?

Apple		Milk	
Apple		Chocolate	
Apple	vs	Ice cream	
Milk		Chocolate	
Milk		Ice cream	
Chocolate		Ice cream	

	comparisons			Preferred item	Notation
1	Apple		Milk	Milk	Apple $\prec$ Milk
2	Apple		Chocolate	Chocolate	Apple $\prec$ Chocolate
3	Apple	vs	Ice cream	Apple	Apple$\prec$Ice cream
4	Milk		Chocolate	Indifferent	Milk $\sim$ Chocolate
5	Milk		Ice cream	*****	----
6	Chocolate		Ice cream	Ice cream	Chocolate $\preccurlyeq$ Ice cream

1. Not complete: # 5
2. Inconsistent: Apple, Chocolate, Ice cream

   Chocolate $\succ$ Apple, Apple $\succ$ Ice cream, Ice cream $\succcurlyeq$ Chocolate

   <u>No transitive preferences</u>

	comparisons			Preferred item	Notation
1	Apple		Milk	Milk	Apple $\prec$ Milk
2	Apple		Chocolate	Chocolate	Apple $\prec$ Chocolate
3	Apple	vs	Ice cream	Apple	Apple $\succ$ Ice cream
4	Milk		Chocolate	Indifferent	Milk $\sim$ Chocolate
5	Milk		Ice cream	I do not know	XXXX
6	Chocolate		Ice cream	Ice cream	Chocolate $\preccurlyeq$ Ice cream

Problems:

1. Not complete: # 5
2. Inconsistent: Apple, Chocolate, Ice cream

   Chocolate $\succ$ Apple, Apple $\succ$ Ice cream, Ice cream $\succcurlyeq$ Chocolate

   <u>No transitive preferences</u>

	comparisons			Preferred item	Notation
1	Apple	vs	Milk	Milk	Apple $\prec$ Milk
2	Apple		Chocolate	Chocolate	Apple $\prec$ Chocolate
3	Apple		Ice cream	Ice cream	Apple $\prec$ Ice cream
4	Milk		Chocolate	Indifferent	Milk $\sim$ Chocolate
5	Milk		Ice cream	Ice cream	Milk $\prec$ Ice cream
6	Chocolate		Ice cream	Ice cream	Chocolate $\prec$ Ice cream

Preferences are:

1. Complete
2. Consistent: Ice cream $\succ$ Chocolate $\sim$ Milk $\succ$ Apple
   <u>Transitive preferences</u>

   When preferences are complete and transitive we say that **preferences are rational**.

Usually we represent preferences using utility functions $u(x)$ where x is an element of the set X (set of items)

In the previous example $\mathrm{X} = \{Apple,\ Milk,\ Chocolate,\ Ice\ cream\}$

A function $u: X \rightarrow R$ is an utility function representing preferences on X if for each pair of items $x, y \in X$

If $x \succcurlyeq y$ if and only if $u(x) \geq u(y)$

Example:

In the previous example:

Ice cream $\succ$ Chocolate $\sim$ Milk $\succ$ Apple

Function $u(x)$ represents these preferences if and only if

$u(Ice\ cream) > u(Chocolate) = u(Milk) > u(Apple)$

i.e.

$u(Ice\ cream) = 4$
$u(Chocolate) = u(Milk) = 3$
$u(Apple) = 2$

Preferences can be represented by an utility function only if they are rational

(necessary condition)

The reverse is not necessarily true: for example lexicographic preferences are rational but cannot be represented by an utility function

We need more assumptions:

For example either a finite set X or continuous preferences

Choice under risk

Up until now, we have thought of the objects that our decision makers are choosing as being physical items

However, we can also think of cases where the outcomes of the choices we make are uncertain - we don't know exactly what will happen when we do a particular choice. For example:

- You are deciding whether or not to invest in a business
- You are deciding whether or not to go skiing next month
- You are deciding whether or not to buy a house that straddles the San Andreas fault line

In each case the outcomes are uncertain.

Here we are going to think about how to model a decision maker who is making such choices.

Economists tend to differentiate between two different types of ways in which we may not know for certain what will happen in the future: ***risk*** and ***ambiguity***.

Risk: the probabilities of different outcomes are known, ***Ambiguity***: the probabilities of different outcomes are unknown

Now we consider models of choice under risk,

An example of choice under risk

For an amount of money ***£ x***, you can flip a coin. If you get heads, you get ***£10***. If you get tails, you get ***£0***.

Assume there is a 50% chance of heads and a 50% chance of tails.

For what price ***x*** would you choose to play the game?

i.e. you have a choice between the following two options.

- 1. Not play the game and get nothing
- 2. Play the game, and get $-x$ for sure, plus a 50% chance of getting $10.

Figure out the expected value (or average pay-out) of playing the game, and see if it is bigger than 0. If it is, then play the game, if not, then don't.

With a 50% chance you will get £$10 - x$,

With a 50% chance you will get $-x$.

Thus, the average payoff is:

$$0.5(10 - x) + 0.5(-x) = 5 - x$$

Thus the value of the game is *£* $5 - x$.

you should play the game if the cost of playing is less than *£ 5*.

Lotteries (or prospects)

Decision making under risk can be considered as a process of choosing between different lotteries.

A lottery (or prospect) consists of a number of possible outcomes with their associated probability

It can be described as:

$$\boldsymbol{q} = (x_1, p_1; x_2, p_2; \ldots x_n, p_n)$$

where

x_i represents the *i^th^* outcome and

p_i is its associated probability, $p_i \in [0,1]\ \forall i$ and $\sum_i p_i = 1$.

In the example the choice is between:

$$\boldsymbol{r} = (10 - x, 0.5; -x, 0.5)$$
$$\boldsymbol{s} = (0, 1)$$

in this last case we omit probability and we can write $\boldsymbol{s} = (0)$.

When an outcomes is for sure (i.e. its probability is 1) we write only the outcome.

$\boldsymbol{s} = (x)$ means that the outcome x is for sure

Sometime we can omit the zero outcomes, so the lottery $\boldsymbol{r} = (10, 0.5; 5, 0.3; 0, 0.2)$ can be written as $\boldsymbol{r} = (10, 0.5; 5, 0.3)$

Compound lotteries

Lotteries can be combined

From the previous example:

suppose you have the following lottery of lotteries: $\boldsymbol{c} = \left(\boldsymbol{r}, \frac{1}{2}; \boldsymbol{s}, \frac{1}{2}\right)$

where

$\boldsymbol{r} = (10 - x, 0.5; -x, 0.5)$ and

$\boldsymbol{s} = (0, 1)$.

Then, the resulting lottery is:

$$\boldsymbol{c} = \left(10 - x, \frac{1}{4};\ -x, \frac{1}{4};\ 0, \frac{1}{2}\right)$$

More in general

Consider the two following lotteries

$\boldsymbol{r} = (x_1, p_1;\ \dots x_n, p_n)$ and

$\boldsymbol{s} = (y_1, q_1;\ \dots y_n, q_n)$,

then

$$\boldsymbol{c} = (\boldsymbol{r}, a; \boldsymbol{s}, 1-a)$$
$$=$$
$$(x_1, ap_1;\ \dots x_n, ap_n; y_1, (1-a)q_1;\ \dots y_n, (1-a)q_n)$$

Expected Value

The expected value of prospect $\boldsymbol{r} = (x_1, p_1;\ \dots x_n, p_n)$ is

$$E(\boldsymbol{r}) = \sum_i p_i \cdot x_i$$

Example

$$\boldsymbol{r} = (1000, 0.25; 500, 0.75)$$
$$E(\boldsymbol{r}) = 0.25 \cdot 1000 + 0.75 \cdot 500$$

St. Petersburg paradox

- A fair coin is tossed repeatedly until a tail appears, ending the game.
- The pot starts at 2 dollars and is doubled every time a head appears.
- Prize is whatever is in the pot after the game ends:
  - 2 dollars if a tail appears on the first toss,
  - 4 dollars if a head appears on the first toss and a tail on the second,
  - 8 dollars if a head appears on the first two tosses and a tail on the third,
  - 16 dollars if a head appears on the first three tosses and a tail on the fourth, etc.
  - 2^k dollars if the coin is tossed *k* times until the first tail appears.

The expected value is ∞:

$$2 \cdot \frac{1}{2} + 4 \cdot \frac{1}{4} + 8 \cdot \frac{1}{8} + \cdots .. =$$
$$= \sum_{i=1}^{\infty} 2^i \cdot \frac{1}{2^i} =$$
$$= 1 + 1 + 1 + \cdots .. = \infty$$

The experimental evidence is that people are willing to pay only limited amount of money to play this lottery

Solution: the value that people attach to the first dollar of their wealth is larger tat the value they attach to the i^{th} dollar they earn.

A decreasing marginal value can explain this paradox

Assumptions on preferences over lotteries

These axioms are related to the axioms on preferences and impose rationality to the individual's behaviour when individuals face choices among lotteries.

$\succcurlyeq$ satisfies:

a. Completeness

For all lotteries $\boldsymbol{q}$ and $\boldsymbol{r}$ we have that $\boldsymbol{q} \succcurlyeq \boldsymbol{r}$ or $\boldsymbol{r} \succcurlyeq \boldsymbol{q}$ (or both)

b. Transitivity

For any three lotteries $\boldsymbol{q}, \boldsymbol{r}, \boldsymbol{s}$ if $\boldsymbol{q} \succcurlyeq \boldsymbol{r}$ and $\boldsymbol{r} \succcurlyeq \boldsymbol{s}$, then $\boldsymbol{q} \succcurlyeq \boldsymbol{s}$

c. Continuity

For any three lotteries $\boldsymbol{q}, \boldsymbol{r}, \boldsymbol{s}$ where $\boldsymbol{q} \succcurlyeq \boldsymbol{r}$ and $\boldsymbol{r} \succcurlyeq \boldsymbol{s}$, there exists some probability p such that there is indifference between the middle ranked prospect $\boldsymbol{r}$ and the prospect $(\boldsymbol{q}, p; \boldsymbol{s}, 1-p)$, i.e.

$$(\boldsymbol{q}, p; \boldsymbol{s}, 1-p) \sim \boldsymbol{r}$$

Equivalently

there exist $a, b \in [0, 1]$ such that:

$$(\boldsymbol{q}, a; \boldsymbol{s}, 1-a) \succcurlyeq \boldsymbol{r} \succcurlyeq (\boldsymbol{q}, b; \boldsymbol{s}, 1-b)$$

d. Independence

Any state of the world that results in the same outcome regardless of one's choice can be ignored or cancelled

For any three lotteries $\boldsymbol{q}, \boldsymbol{r}, \boldsymbol{s}$ and any $p \in [0, 1]$

if $\boldsymbol{q} \succcurlyeq \boldsymbol{r}$

then $(\boldsymbol{q}, p; \boldsymbol{s}, 1-p) \succcurlyeq (\boldsymbol{r}, p; \boldsymbol{s}, 1-p)$

Example

If $\boldsymbol{q} = (3000)$, $\boldsymbol{r} = (4000, 0.8)$ and $\boldsymbol{q} \succcurlyeq \boldsymbol{r}$

then

$\boldsymbol{q}' = (3000, 0.25)$, $\boldsymbol{r}' = (4000, 0.2)$ and $\boldsymbol{q}' \succcurlyeq \boldsymbol{r}'$

Note that:

prospect $\boldsymbol{q}'$ is the compound lottery $\boldsymbol{q}' = (\boldsymbol{q}, 0.25; 0, 0.75)$ and

prospects $\boldsymbol{r}'$ is the compound lottery $\boldsymbol{r}' = (\boldsymbol{r}, 0.25; 0, 0.75)$

Expected Utility

The utility function over lotteries has an *Expected utility form* if for a prospect $\boldsymbol{r} = (x_1, p_1;\ \dots x_n, p_n)$ is given by:

$$U(\boldsymbol{r}) = \sum_i p_i \cdot u(x_i)$$

where $u(x_i)$ is the utility function over outcome x_i

An utility function with expected utility form is called *von Neumann-Morgenstern expected utility function*

Example

$\boldsymbol{r} = (1000, 0.25; 500, 0.75)$ and $u(x_i) = \sqrt{x_i}$

$U(\boldsymbol{r}) = 0.25\sqrt{1000} + 0.75\sqrt{500}$

Representation theorem

Let be X the set of all possible lotteries.

If the preferences over these lotteries are rational (complete and transitive) and satisfy continuity and independence, then there exists a *von Neumann-Morgenstern expected utility function* $U(x)$ such that:

$$q \succcurlyeq r$$

if and only if

$$U(q) \geq U(r)$$

Risk Attitudes

A decision maker is ***risk neutral*** if he is indifferent between receiving a lottery's expected value and playing the lottery.

Consider $\boldsymbol{r} = (x_1, p_1;\ \ldots x_n, p_n)$ then:

$$u\left(\sum_i p_i \cdot x_i\right) = \sum_i p_i \cdot u(x_i)$$

A decision maker is risk neutral if its utility function is linear, i.e. $u(x) = a + b\ x$

Example:

Lottery $r = (10, 0.6; 2, 0.4)$
Its expected value is $E(r) = 10 \cdot 0.6 + 2 \cdot 0.4 = 6.8$

Risk neutral agent is indifferent between receiving (and playing) lottery r and receiving 6.8 for sure:

$$u(6.8) = 0.6 \cdot u(10) + 0.4 \cdot u(2)$$

A decision maker is ***risk averse*** if he prefers receiving the lottery's expected value instead of playing the lottery.

Consider $\boldsymbol{r} = (x_1, p_1;\ \dots x_n, p_n)$ then:

$$u\left(\sum_i p_i \cdot x_i\right) > \sum_i p_i \cdot u(x_i)$$

A decision maker is risk averse if its utility function is strictly concave, i.e. $u"(x) < 0$

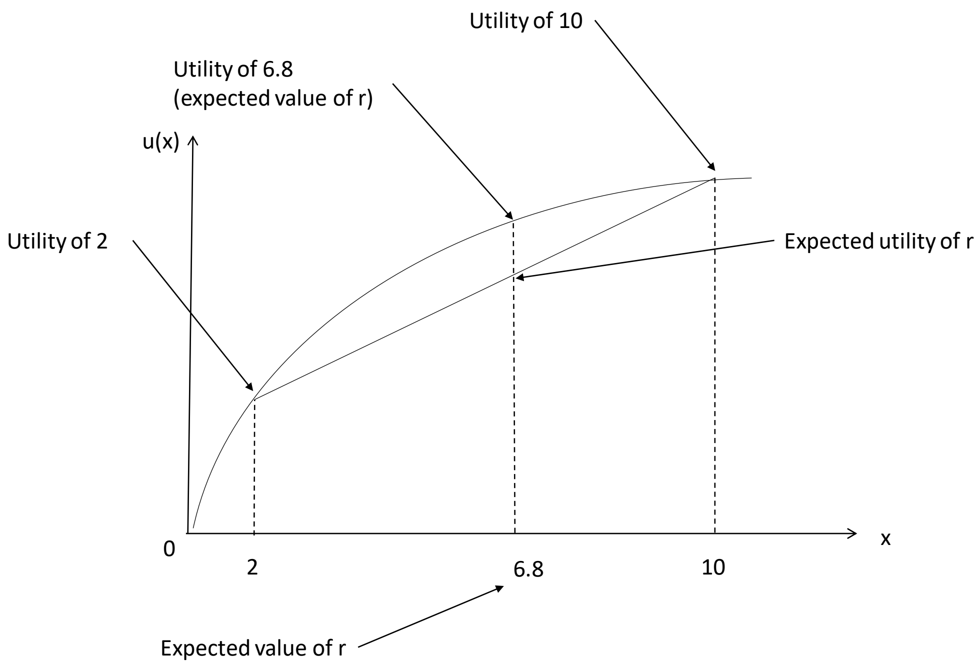
Example:

Lottery $r = (10, 0.6; 2, 0.4)$

Its expected value is $E(r) = 10 \cdot 0.6 + 2 \cdot 0.4 = 6.8$

Risk averse agent prefers receiving 6.8 for sure that receiving (and playing) lottery r

$$u(6.8) > 0.6 \cdot u(10) + 0.4 \cdot u(4)$$



A decision maker is ***risk seeking (or risk lover)*** if he prefers playing the lottery instead of receiving its expected value.

Consider $\boldsymbol{r} = (x_1, p_1;\ \dots x_n, p_n)$ then:

$$u\left(\sum_i p_i \cdot x_i\right) < \sum_i p_i \cdot u(x_i)$$

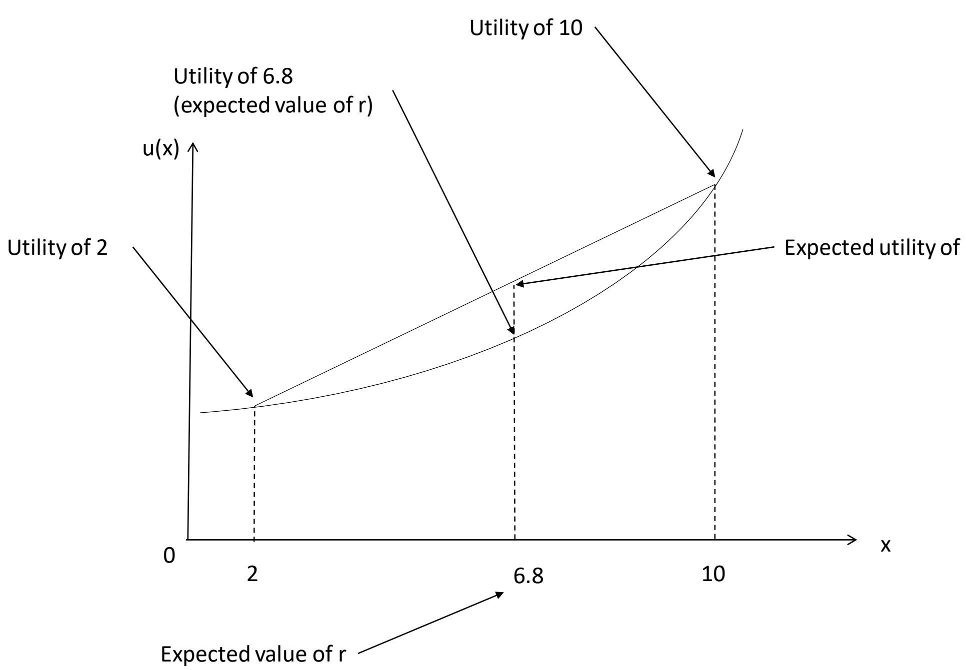
A decision maker is risk seeking if its utility function is strictly convex, i.e. $u''(x) > 0$

Example:

Lottery $r = (10, 0.6; 2, 0.4)$
Its expected value is $E(r) = 10 \cdot 0.6 + 2 \cdot 0.4 = 6.8$

Risk lover agent prefers receiving (and playing) lottery r that receiving 6.8 for sure

$$u(6.8) < 0.6 \cdot u(10) + 0.4 \cdot u(4)$$



All these results are proved by Jensen's Inequality

Let x be a random variable where $E(x)$ is its expected value and $f(x)$ is a concave function then:

$$f\big(E(x)\big) \geq E\big(f(x)\big)$$

$f(x)$ is a convex function then:

$$f\big(E(x)\big) \leq E\big(f(x)\big)$$

Measures of risk aversion

For of a lottery $\boldsymbol{q}$, the risk premium $R(\boldsymbol{q})$ is defined as

$$R(\boldsymbol{q}) = E(\boldsymbol{q}) - CE(\boldsymbol{q})$$

where $CE(\boldsymbol{q})$ is the *certainty equivalent wealth* defined as

$$u\big(CE(\boldsymbol{q})\big) = U(\boldsymbol{q})$$

Interpretation:

the risk premium $R(\boldsymbol{q})$ is the amount of money that an agent is willing to pay to eliminate the risk.

Example.

Person A has to "play" the following lottery $\boldsymbol{q} = (100\ \$, 0.5; 64\ \$, 0.5)$.

Assume that his utility function is $u(x) = \sqrt{x}$

Compute the risk premium $R(\boldsymbol{q})$.

$$u\big(CE(\boldsymbol{q})\big) = U(\boldsymbol{q})$$

$$\sqrt{CE(\boldsymbol{q})} = 0.5\sqrt{100} + 0.5\sqrt{64}$$

$$CE(\boldsymbol{q}) = 81$$

$$E(\boldsymbol{q}) = 100 \cdot 0.5 + 64 \cdot 0.5 = 82$$

$R(\boldsymbol{q}) = E(\boldsymbol{q}) - CE(\boldsymbol{q}) = 82 - 81 = 1\ \$$

Person B utility function is $u(x) = x$. He proposes to person A to buy the lottery. Which is the minimum price that person A will accept?

Answer: 81 $

Is convenient for person B?

$u(r) = 0.5 \cdot 100 + 0.5 \cdot 64 = 82 > u(81) = 81$

Answer: yes

Selling the lottery for 81 $ is equivalent to hold the lottery and pay 1 $ to player B that agrees to pay 18 $ in the bad state (when the outcome is 64 $) and to receive 18 $ in the good state (when the outcome is 100 $)

For 1 $ player B bears all the risk

1. Arrow-Pratt measure of absolute risk-aversion:
$$A(c) = -\frac{u"(c)}{u'(c)}$$

2. Arrow-Pratt-De Finetti measure of relative risk-aversion or coefficient of relative risk aversion
$$R(c) = -\frac{c \cdot u"(c)}{u'(c)}$$

Type of Risk-Aversion	Example of utility functions
Increasing absolute risk-aversion	$u(w) = -e^{-w^2}$
Constant absolute risk-aversion	$u(w) = -e^{-c \cdot w}$
Decreasing absolute risk-aversion	$u(w) = \ln w$

Type of Risk-Aversion	Example of utility functions
Increasing relative risk-aversion	$w - cw^2$
Constant relative risk-aversion	$\ln(w)$
Decreasing relative risk-aversion	$-e^{2 \cdot w^{-\frac{1}{2}}}$