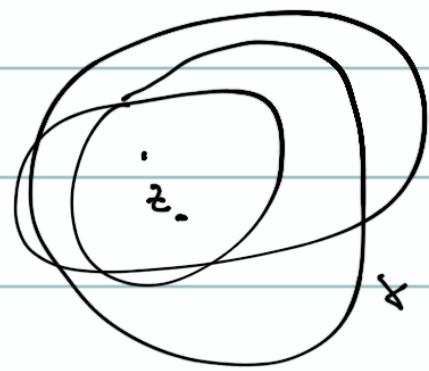
$$\frac{1}{z-z_0}\ , \qquad \frac{1}{(z-z_0)^n} \qquad n=2,3,\dots$$

$$\frac{d}{dz}\left(\frac{1}{z-z_0}\right) = -\frac{1}{(z-z_0)^2}$$

Se γ è una curva chiusa che circonda z_0



$$\int_\gamma \frac{1}{z-z_0}\,dz = 2\pi i \, \mathrm{Ind}_\gamma(z_0)$$

Se $n \geq 2$

$$\int_\gamma \frac{dz}{(z-z_0)^n} = 0$$

Sia $f \in H(U \setminus \{z_0\})$, U intorno di z_0.

Def. z_0 si dice singolarità rimovibile (eliminabile) per f se f si può estendere a tutto $\mathcal{U}$ come funz. olomorfa.

Es. Sia $h : H(\mathcal{U})$ e poniamo

$$f(z) = \boxed{\frac{h(z)-h(z_0)}{z-z_0}} \in H(\mathcal{U}\setminus\{z_0\})$$

estendendo f a z_0 ponendo:

$$f(z_0) = h'(z_0)$$

si ottiene una funz. olomorfa su tutto $\mathcal{U}$.

<u>Teor.</u> Sia $f \in H(\Omega \setminus \{z_0\})$, $z_0 \in \Omega$.

Se $\exists\, M > 0$ t.c.

$$\underline{|f(z)| \le M} \quad \forall z \in B_r(z_0)\setminus\{z_0\}$$

dove $r>0$ è t.c.

$$B_r(z_0) \subset \Omega$$

allora z_0 è una sing. rimovibile (per f).

Dim. Sia

$$g(z) = \begin{cases} \underline{\underline{(z-z_0)^2 f(z)}} & , z \in \Omega \setminus \{z_0\}, \\ 0 & , z = z_0 . \end{cases}$$

$g \in H(\Omega \setminus \{z_0\})$. g è diff. in senso complesso in z_0: $\forall\, z \neq z_0$

$$\frac{g(z) - g(z_0)}{z - z_0} = \frac{(z-z_0)^{\cancel{2}} f(z)}{\cancel{z - z_0}} \underset{z \to z_0}{\longrightarrow} 0 .$$

Per il Teor. di Goursat $g \in H(\Omega)$.

$g(z_0) = 0$, $g'(z_0) = 0$

$$g(z) = \sum_{k=2}^{\infty} \underbrace{\frac{g^{(k)}(z_0)}{k!}}_{c_k} (z - z_0)^k =$$

$$= \underline{(z-z_0)^2} \sum_{k=2}^{\infty} c_k (z - z_0)^{k-2}$$

per $z \neq z_0$

$$f(z) = \frac{g(z)}{(z-z_0)^2} = \sum_{k=2}^{\infty} c_k (z-z_0)^{k-2}$$

$$\lim_{z \to z_0} f(z) = c_2$$

Ponendo $f(z_0) = c_2$

f si estende a una funz. olomorfa su tutto Ω. $\square$

Teor. Sia $f \in H(\Omega \setminus \{z_0\})$, $z_0 \in \Omega$.

Si danno 3 casi:

1) z_0 è sing. rimovibile.

2) Esistono numeri $c_1, \dots, c_m \in \mathbb{C}$ t.c.

ordine del polo

$$f(z) - \sum_{k=1}^{m} \frac{c_k}{(z-z_0)^k}$$ ha una sing. rimovibile.

3) $\forall r > 0$ t.c. $B_r(z_0) \subset \Omega$, $f(B_r(z_0) \setminus \{z_0\})$ è denso in $\mathbb{C}$.

Def. Se vale il caso 2) si dice che z_0 è un polo (sing. polare) per f.

Se vale il caso 3) si dice che z_0 è una sing. essenziale per f.

Dim. Supp. che non valga 3). Esiste $r > 0$ t.c. $B_r(z_0) \subset \Omega$ e

$$f(B_r(z_0) \setminus \{z_0\}) \text{ non è denso in } \mathbb{C}.$$

cioè $\exists\, w \in \mathbb{C}$ e $\delta > 0$ t.c.

$$|f(z) - w| \geq \delta \quad \forall\, z \in B_r(z_0) \setminus \{z_0\}.$$

Sia $\boxed{g(z) = \dfrac{1}{f(z) - w}}$, $z \in B_r(z_0) \setminus \{z_0\}$.

$$g \in H(B_r(z_0) \setminus \{z_0\})$$

$$|g(z)| \leq \frac{1}{\delta} \quad \forall\, z \in B_r(z_0) \setminus \{z_0\}$$

Quindi z_0 è rimov. per g.

Se $\underline{\underline{g(z_0) \neq 0}}$ $\qquad f(z) - w = \dfrac{1}{g(z)}$ per z suff. vicino a z_0. $\quad f(z) = w + \dfrac{1}{g(z)} \in H(B_\rho(z_0)')$.

Dove ρ è t.c. $0 < \rho \leq r$. In questo caso z_0 è rimovibile per f (Caso 1)).

Se $g(z_0) = 0$, $\exists\ m = 1, 2, \dots$ t.c.

$$g(z) = \sum_{k=m}^{\infty} \frac{g^{(k)}(z_0)}{k!} (z - z_0)^k$$

con $\qquad g^{(m)}(z_0) \neq 0$

$$g(z) = \underline{(z - z_0)^m} \underbrace{\sum_{j=0}^{\infty} b_j (z - z_0)^j}_{h(z)}, \quad b_0 \neq 0.$$

$h(z_0) = b_0 \neq 0$.

Allora $\exists\ \rho$, $0 < \rho \leq r$ t.c. $\quad g(z) \neq 0$

$\forall\ z \in B_\rho(z_0) \setminus \{z_0\}$

$$f(z) - w = \frac{1}{g(z)} = \frac{1}{(z - z_0)^m} \frac{1}{h(z)} =$$

$$= \frac{1}{(z - z_0)^m} \left(\frac{1}{b_0} + \sum_{j \geq 1} d_j (z - z_0)^j \right) =$$

$$= \frac{\frac{1}{b_0}}{(z-z_0)^m} + \frac{d_1}{(z-z_0)^{m-1}} + \dots + d_m +$$

$$+ d_{m+1}(z-z_0)^1 + \dots + d_j (z-z_0)^{\overbrace{j-m}^{>0}} + \dots$$

$$c_m = \frac{1}{b_0}, \quad c_{m-1} = d_1, \dots, c_0 = d_m$$

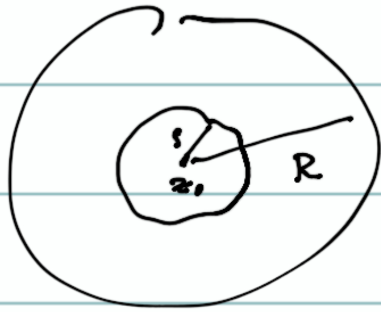
$$\Rightarrow f(z) = \underbrace{\sum_{j=0}^{m} \frac{c_j}{(z-z_0)^j}} + \underbrace{d_m \xrightarrow{+W} + d_{m+1}(z-z_0) + \dots}_{\text{olomorfa in tutta } B_\rho(z_0)}$$

$c_m = \frac{1}{b_0} \neq 0$!!

Quindi si ricade nel Caso 2) di una sing. polare. □

Es. $\underline{f(z) = e^{\frac{1}{z}}}$, $z_0 = 0$ è sing. essenziale.

$$f(z) = \sum_{k=0}^{\infty} \frac{1}{k!} \frac{1}{z^k}$$



$f \in H(B_R(z_0) \setminus \overline{B_\rho(z_0)})$

si sottintende che se

$\rho = 0$ $\quad \overline{B_\rho(z_0)} \sim \{z_0\}$.

<u>Lemma</u> Sia $g \in H(B_R(z_0) \setminus \overline{B_\rho(z_0)})$ poniamo

$\forall \quad r \in (\rho, R)$

$$a(r) = \int_{\partial B_r(z_0)} g(z)\, dz$$

$a(r)$ è costante, cioè è indip. da r.

<u>Dim 1.</u> Siano r_1, r_2 $\quad \rho < r_1 < r_2 < R$

$$G = B_{r_2}(z_0) \setminus \overline{B_{r_1}(z_0)}, \quad \overline{G} \subset B_R(z_0) \setminus \overline{B_\rho(z_0)}$$

e ∂G è regolare.

$$\int_{\partial G} g(z)\, dz = 0$$

$$0 = \int_{\partial G} g(z)\, dz = \int_{\partial B_{r_2}(z_0)} g(z)\, dz - \int_{\partial B_{r_1}(z_0)} g(z)\, dz =$$

$$= a(r_2) - a(r_1)$$

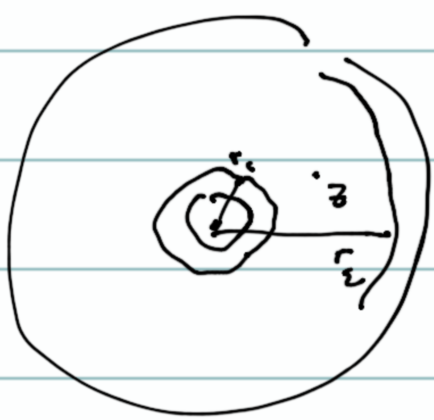
$$a(r_2) = a(r_1) \quad \forall \ r_1, r_2 \qquad \square$$

<u>Dim 2.</u> $$a(r) = \int_0^{2\pi} g(re^{i\theta})\, r i e^{i\theta}\, d\theta$$

per esercizio!

$$\frac{d}{dr} a(r) = \cdots = 0 \qquad \square$$

Sia $f \in H\left(B_R(z_0) \setminus \overline{B_\rho(z_0)}\right)$ sia z un punto nella corona, $\exists \ r_1, r_2$ t.c.



$$\rho < r_1 < |z - z_0| < r_2 < R$$

$$B_{r_2}(z_0) \setminus \overline{B_{r_1}(z_0)}$$

Per la formula di Cauchy

$$f(z) = \underbrace{\frac{1}{2\pi i} \int_{\partial B_{r_2}(z_0)} \frac{f(\xi)}{\xi - z}\, d\xi}_{\mathrm{I}} - \underbrace{\frac{1}{2\pi i} \int_{\partial B_{r_1}(z_0)} \frac{f(\xi)}{\xi - z}\, d\xi}_{\mathrm{II}}.$$

$$(I) = \frac{1}{2\pi i} \int_{\partial B_{r_2}(z_0)} \frac{f(\xi)}{\xi - z} \, d\xi$$

$$\xi - z = (\xi - z_0) - (z - z_0) = (\xi - z_0)\Big(1 - \underbrace{\frac{z - z_0}{\xi - z_0}}_{p}\Big)$$

$$p = \frac{z - z_0}{\xi - z_0}, \quad |p| < 1$$

$$\frac{1}{\xi - z} = \frac{1}{(\xi - z_0)} \left(\frac{1}{1 - p} \right) = \frac{1}{\xi - z_0} \sum_{k=0}^{\infty} p^k$$

$$(I) = \frac{1}{2\pi i} \int_{\partial B_{r_2}(z_0)} f(\xi) \frac{1}{\xi - z_0} \sum_{k=0}^{\infty} \frac{(z - z_0)^k}{(\xi - z_0)^k} \, d\xi$$

$$= \frac{1}{2\pi i} \sum_{k=0}^{\infty} \boxed{\int_{\partial B_{r_2}(z_0)} \frac{f(\xi)}{(\xi - z_0)^{k+1}} \, d\xi} \, (z - z_0)^k$$

Posto $a_k = \int_{\partial B_r(z_0)} \frac{f(\xi)}{(\xi - z_0)^{k+1}} \, d\xi$, $\rho < r < R$

$$\boxed{(I) = \frac{1}{2\pi i} \sum_{k=0}^{\infty} a_k (z - z_0)^k} \, .$$

$$(\mathrm{II}) = \frac{1}{2\pi i} \int_{\partial B_{r_1}(z_0)} \frac{f(\xi)}{\xi - z}\, d\xi$$

$$\xi - z = \underbrace{(\xi - z_0)} - \underbrace{(z - z_0)}$$

maggiore in modulo

$$\xi - z = (z - z_0)\left(\frac{\xi - z_0}{z - z_0} - 1\right) = -(z - z_0)(1 - p)$$

p

$$(\mathrm{II}) = -\frac{1}{2\pi i} \int_{\partial B_{r_1}(z_0)} f(\xi)\, \frac{1}{(z - z_0)}\, \frac{1}{1 - p}\, d\xi =$$

$$= -\frac{1}{2\pi i} \int_{\partial B_{r_1}(z_0)} f(\xi)\, \frac{1}{(z - z_0)} \sum_{j=0}^{\infty} \frac{(\xi - z_0)^j}{(z - z_0)^j}\, d\xi =$$

$$= -\frac{1}{2\pi i} \sum_{j=0}^{\infty} \int_{\partial B_{r_1}(z_0)} f(\xi)(\xi - z_0)^j\, d\xi\, \frac{1}{(z - z_0)^{j+1}}.$$

$n = -(j+1)$, $j + n = -1$, $j = -n - 1$

$$= -\frac{1}{2\pi i} \sum_{n=-\infty}^{-1} \underbrace{\int_{\partial B_{r_1}(z_0)} f(\xi)(\xi - z_0)^{-n-1}\, d\xi}_{a_n}\, (z - z_0)^n$$

$$(\mathrm{II}) = -\frac{1}{2\pi i}\sum_{n=-\infty}^{-1} a_n (z-z_0)^n$$

$$f(z) = (\mathrm{I}) - (\mathrm{II}) = \frac{1}{2\pi i}\sum_{n=-\infty}^{+\infty} a_n (z-z_0)^n$$

$$a_n = \int_{\partial B_r(z_0)} f(\zeta)(\zeta - z_0)^{-n-1}\, d\zeta$$

$$\forall\, r \in (\rho, R).$$