Exercises QFT II — 2018/2019

Problem Sheet 5

Problem 9: On-shell renormalization prescription

Consider the renormalized 4pt function for the scalar field theory with quartic potential in the (modified) minimal subtraction scheme ($\overline{MS}$):

$$
\begin{aligned}
i\tilde{\Gamma}_r^{(4)}(s,t,u) \;&=\; -i\lambda_r - i\frac{\lambda_r^2}{32\pi^2}\int_0^1 dx\,\left[\ln\left(\frac{m^2 - x(1-x)s}{4\pi\mu^2}\right)\right. \\
&\quad \left. + \ln\left(\frac{m^2 - x(1-x)t}{4\pi\mu^2}\right) + \ln\left(\frac{m^2 - x(1-x)u}{4\pi\mu^2}\right)\right] + \mathcal{O}(\lambda_r^3)
\end{aligned} \tag{1}
$$

The Mandelstam variables must satisfy the following condition on-shell: $s + t + u = 4m_{\text{ph}}^2$, where m_{ph} is the physical mass.

1. Explain why we can substitute m with m_{ph} at this order in perturbation theory.

Define the physical coupling constant to be

$$
\lambda_{\text{ph}} \equiv -\tilde{\Gamma}^{(4)}(s = 4\bar{E}^2, t = u = 2m^2 - 2\bar{E}^2)\,, \tag{2}
$$

where $\bar{E}$ is some reference energy scale (at which λ_{ph} is measured).

2. Compute λ_{ph}.

3. Express λ_r in terms of λ_{ph}.

4. Express $\tilde{\Gamma}_r^{(4)}$ in terms of λ_{ph} (substitute the expression of λ_r found in the previous point). The dependence on the arbitrary scale μ should disappear.

5. By studying the kinematics of the $2 \to 2$ scattering process, interpret $\bar{E}$ as the energy of the colliding particle in the Center of Mass system.

Problem 10: $\lambda\phi^3$ theory in 6 dimensions

Consider the interacting scalar theory with Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \frac{m^2}{2}\phi^2 - \frac{\lambda}{6}\phi^3 \,. \tag{3}$$

1. Show that when the number of spacetime dimensions is 6, the coupling constant λ is dimensionless.
2. Derive the Feynman rules.
3. Draw the tree-level and one-loop graphs for the two-point function.
4. Applying the Feynman rules, show that

   $$\tilde{\Gamma}^{(2)} = p^2 - m^2 + \Pi \tag{4}$$

   where, in 6 dimensions,

   $$\Pi = i\alpha\lambda^2 \int \frac{d^6q}{(2\pi)^6} \frac{i}{q^2 - m^2 + i\epsilon} \frac{i}{(p-q)^2 - m^2 + i\epsilon} \tag{5}$$

   and the constant α should be determined.
5. Is the integral above convergent?
6. Use dimensional regularization and compute Π isolating the divergent term in 6 dimensions. You should get a result of the form

   $$\frac{1}{3-\omega}(A\,m^2 + B\,p^2) \tag{6}$$

7. Find the expression for the physical mass m_{ph} (neglecting the terms that vanish when $\omega \to 3$).
8. How can we reabsorb the divergent term proportional to p^2? Notice that we still have the possibility of scaling the field ϕ by an arbitrary constant $Z_\phi^{1/2}$ that can depend on ω, i.e.

   $$\phi = Z_\phi^{1/2}\phi_r \,. \tag{7}$$

   Find the constant Z_ϕ such that

   $$\tilde{\Gamma}_r^{(2)} = p^2 - m_{\text{ph}}^2 + \mathcal{O}(\lambda) \,, \tag{8}$$

   where $\Gamma_r^{(2)} \sim \langle\phi_r\phi_r\rangle^{-1}$ is the two-point function computed by using the renormalized field ϕ_r instead of ϕ.