1.1.1 The continuum hypothesis and the concept of material element

(subsec-1)

What is a continuum?

Fluid mechanics is concerned with the behaviour of matter on a macroscopic scale which is large with respect to the distance between molecules whose structure does not need to be taken into account explicitly. The behaviour of fluids is assumed to be the same as they were perfectly continuous in structure and the physical properties of the matter contained within a given small volume are regarded as being spread uniformly over that volume. This is the so called *continuum hypothesis*, which is supposed to be hereafter valid without exceptions. From the observational viewpoint, the reason why the particle structure of the fluid is irrelevant is that the sensitive volume of a certain instrument embedded in the fluid itself is small enough for the measurement to be a local one relative to the macroscopic scale even if it is large enough for the fluctuations arising from the molecular motion to have no effect on the observed average. If the volume of fluid to which the instrument responds were comparable with the volume in which variations due to molecular fluctuations take place, observations would fluctuate from one observation to another and the results would vary in an irregular way with the size of the sensitive volume of the instrument. On the opposite, if the volume of fluid to which the instrument responds were too large relatively to the variations associated to spatial distribution of physical quantities, the instrument would not be able to detect even macroscopic features of the fluid. Therefore, the fluid is regarded as a continuum when the measured fluid property is constant for sensitive volumes small on the macroscopic scale, but large on the microscopic (molecular) scale. This concept is illustrated in Fig. 1.1. In any case, there is observational evidence that the common real fluids, both gases and liquids, move as they are continuous both under normal conditions and also for considerable departures from them.

The concept of material element

(Sec1.1) Physical laws apply directly to a fixed collection of matter and for this reason the equations governing fluid mechanics and thermodynamics are developed most intuitively in a framework where the dynamical or physical quantities refer to identifiable pieces of matter (*Lagrangian description*) rather than to fields (*Eulerian description*). The former description relies on the concept of *material volume* of fluid (or *parcel*) which, by definition, consists always of the same fluid portion and move with them, as Fig. 1.2 shows. In this perspective, the flow quantities are defined as functions of time and of the choice of a material element of fluid and describe the history of this selected fluid element.

Material elements of fluid change their shape as they move, so each selected element should be selected in such a way that its linear extension is not involved; therefore the element is specified by the position of its centre of mass at some initial instant, on the understanding that its initial linear dimensions are so small as to guarantee smallness at all subsequent instants in spite of distortions and extensions of the element. The Eulerian description is related to the Lagrangian description by the following kinematic constraint: the field property at a given location and time must equal the property possessed by the material element occupying that position at that instant.

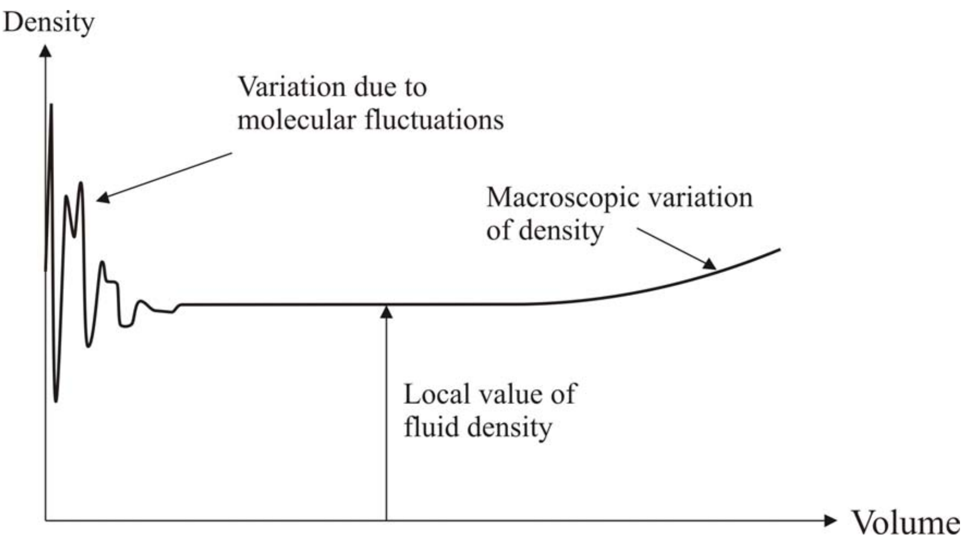

Fig. 1.1 The plot illustrates qualitatively the effect of size of sensitive volume of an instrument on the density measurement.
(Fig-1-1)

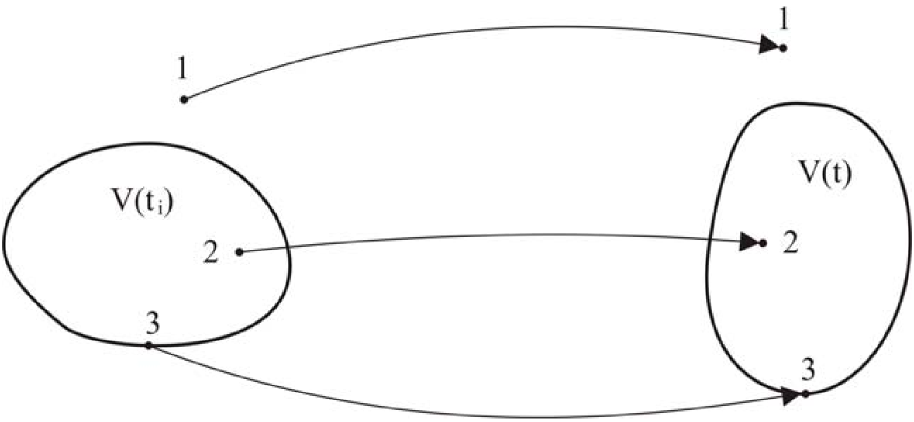

Fig. 1.2 Time evolution of a material volume V of fluid. Assume that, at a certain time t_{i}, point "1" lies outside $V(t_{\mathrm{i}})$, point "2" lies inside $V(t_{\mathrm{i}})$, while point "3" belongs to the surface enclosing $V(t_{\mathrm{i}})$. Then, at each subsequent time t, point "1" will be situated outside $V(t_{\mathrm{i}})$, point "2" will be situated inside $V(t_{\mathrm{i}})$ and point "3" will belong to the surface enclosing $V(t_{\mathrm{i}})$.
(Fig-1-2)

1.1.2 Kinematics of material elements

(subsec-2)

The Lagrangian derivative

A material element of fluid is identified by the "initial" position of its centre of mass, say $\mathbf{x}_0 = (x_0, y_0, z_0)$ in a Cartesian coordinate frame, and by the trajectory $\mathbf{x}(t) = (x(t), y(t), z(t))$ traced out in the course of its motion with velocity $\mathbf{u}(t) = (u, v, w)$, where

$$(2.1) \quad u = \frac{\mathrm{d}x}{\mathrm{d}t}, \qquad v = \frac{\mathrm{d}y}{\mathrm{d}t}, \qquad w = \frac{\mathrm{d}z}{\mathrm{d}t}. \tag{1.1}$$

The vector $\mathrm{d}\mathbf{x} = (\mathrm{d}x, \mathrm{d}y, \mathrm{d}z)$ is the incremental displacement of the given material element during the time interval $\mathrm{d}t$. By the kinematic constraint reported at the end of Section 1.1.1, velocity (1.1) is equal to the field value $\mathbf{u}(\mathbf{x}(t))$ located at the material element's position $\mathbf{x}(t)$ at time t. The time growth rate of a scalar, say θ , representing a physical quantity of a certain material element of fluid in motion depends both on the change of the position of the same fluid element in the course of time and on the explicit change in time of the scalar itself. The situation is playfully described in Fig. 1.3.

Therefore, if δt is a small time interval, the total variation of θ during this interval is

$$(2.2) \quad \theta(\mathbf{x} + \mathbf{u}\,\delta t, t + \delta t) - \theta(\mathbf{x}, t) \tag{1.2}$$

As $\delta t \to 0$, Taylor's expansion of (1.2) yields

$$(2.3) \qquad \theta(\mathbf{x} + \mathbf{u}\,\delta t, t + \delta t) - \theta(\mathbf{x}, t) \tag{1.3}$$

$$= \frac{\partial \theta}{\partial x} u\,\delta t + \frac{\partial \theta}{\partial y} v\,\delta t + \frac{\partial \theta}{\partial z} w\,\delta t + \frac{\partial \theta}{\partial t}\delta t + O\left((\delta t)^2\right) \tag{1.4}$$

$$= \delta t \left(\frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla \right) \theta + O\left((\delta t)^2\right) \tag{1.5}$$

where the components of $\mathbf{u}$ are given by (1.1). For $\delta t \to 0$, the time growth rate of $\theta(\mathbf{x}, t)$ is

$$(2.4) \quad \frac{\theta(\mathbf{x} + \mathbf{u}\,\delta t, t + \delta t) - \theta(\mathbf{x}, t)}{\delta t} = \left(\frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla \right) \theta(\mathbf{x}, t) + O(\delta t) \tag{1.6}$$

and converges to the so-called *Lagrangian derivative* of θ. In other words,

$$(2.5) \quad \lim_{\delta t \to 0} \frac{\theta(\mathbf{x} + \mathbf{u}\,\delta t, t + \delta t) - \theta(\mathbf{x}, t)}{\delta t} = \frac{\mathrm{D}}{\mathrm{D}t} \theta(\mathbf{x}, t) \tag{1.7}$$

where, by definition,

$$\frac{\mathrm{D}}{\mathrm{D}t} := \frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla$$

Thus, the Lagrangian derivative of θ, i.e.

$$(2.6) \quad \frac{\mathrm{D}\theta}{\mathrm{D}t} := \frac{\partial \theta}{\partial t} + \mathbf{u} \cdot \nabla \theta \tag{1.8}$$

includes two contributions:

- the first, $\partial\theta/\partial t$, is introduced by temporal changes at the position $\mathbf{x}$ where the material volume is instantaneously located at time t. It is called the *Eulerian (or local) derivative*;
- the second, $\mathbf{u} \cdot \nabla\theta$, is due to the motion of the material volume to positions with different values of θ. It is the so called *advective term*.

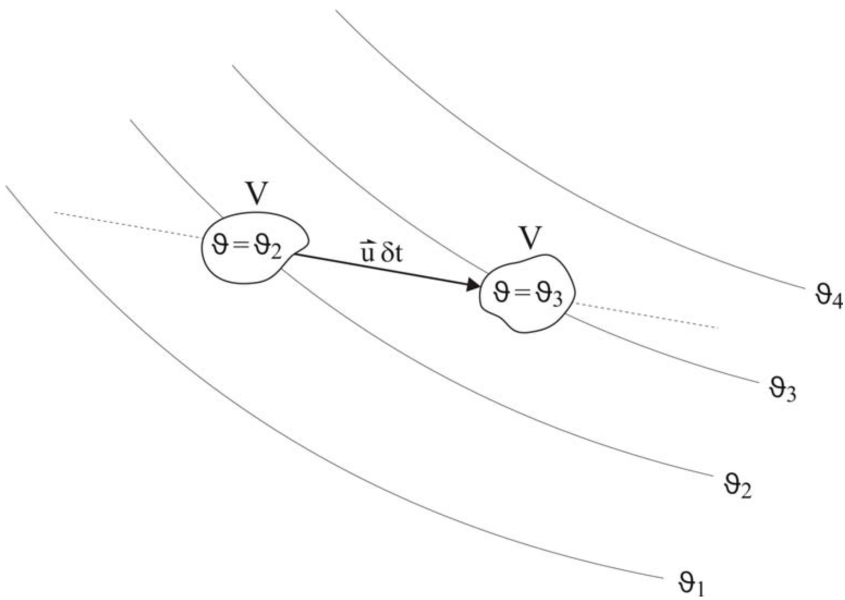

Fig. 1.3 Mr. Lagrange is situated inside the small material volume V, in motion with velocity $\mathbf{u}$ in a portion of space where the scalar field $\theta(\mathbf{x},t)$ is defined. He measures θ and finds the value θ_2 when the material volume that carries him crosses the isoline $\theta = \theta_2$. Then, after a time δt, the same volume displaces itself of the amount $\mathbf{u}\,\delta t$, thus crossing the isoline $\theta = \theta_3$. Mr. Lagrange repeats the measurement and now he finds the value θ_3. ... Finally, he states equation (1.8).
(Fig-2-1)