1.7 The Lagrangian derivative

(a.k.a. the convective derivative, or the material derivative).

We know how to measure the time derivative of a physical quantity associated with the fluid, for example that of the temperature $T(\mathbf{x}, t)$, at a fixed point in space (the Eulerian derivative). It's just

$$\frac{\partial T}{\partial t}.$$

This quantity will describe the change of temperature at a fixed location, for example air temperature in Bristol. However, this quantity will not be a measure of how a mass of air heats up or becomes colder. The reason is that air is swept away by the prevailing flow field $\mathbf{u}(\mathbf{x}, t)$. In other words, to describe the change of temperature of a piece of air, we need to consider the rate of change of $T(\mathbf{x}, t)$, following a fluid particle with trajectory $\mathbf{x}(\mathbf{a}, t)$. This is called the Lagrangian derivative, and is denoted as $\frac{DT}{Dt}$.

Thus using the chain rule, we have

$$\frac{DT}{Dt} \equiv \frac{dT}{dt} = \frac{\partial T}{\partial t} + (\dot{\mathbf{x}} \cdot \nabla)T.$$

But according to (2) and (3), $\dot{\mathbf{x}} = \mathbf{v}(\mathbf{a}, t) = \mathbf{u}(\mathbf{x}, t)$, and so

$$\frac{DT}{Dt} = \frac{\partial T}{\partial t} + (\mathbf{u} \cdot \nabla)T. \tag{7}$$

A particularly important example is the change in velocity (the acceleration) of a fluid particle, which we need to apply Newton's equations to fluid motion. Of course, the velocity is a vector quantity, which means we have to apply (7) to each component:

$$\frac{D\mathbf{u}}{Dt} = \left(\frac{Du_1}{Dt}, \frac{Du_2}{Dt}, \frac{Du_3}{Dt}\right).$$

The the acceleration of a fluid particle becomes

$$\frac{D\mathbf{u}}{Dt} = \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u}. \tag{8}$$

Note: The Lagrangian derivative (i.e. following fluid particles) is given in terms of Eulerian (i.e. fixed point) measurements. It is vital to understand exactly how to compute expressions like $(\mathbf{u} \cdot \nabla)\mathbf{u}$ for a given velocity.

Example: Consider an accelerating fluid flow, such as the log flowing through a narrowing channel. Suppose

$$\mathbf{u} = (U + kx, U - ky, 0).$$

Then

- The flow is steady since $\frac{\partial \mathbf{u}}{\partial t} = \mathbf{0}$.

- The advective term is

$$(\mathbf{u}\cdot\nabla)\mathbf{u} = \left((U+kx)\frac{\partial}{\partial x} + (U-ky)\frac{\partial}{\partial y}\right)(U+kx, U-ky, 0).$$

  Hence the acceleration of the log is

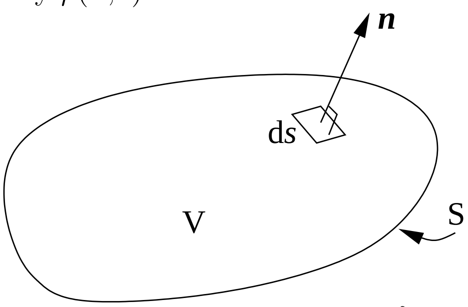
$$\frac{D\mathbf{u}}{Dt} = (k(U+kx), -k(U-ky), 0)\,.$$

1.8 Mass conservation

One of the fundamental laws of continuum mechanics is the law of mass conservation. That is, fluid is neither created or destroyed.

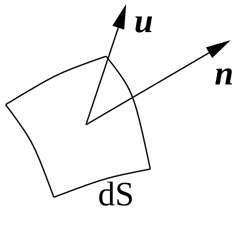
Consider an arbitrary finite volume, V, which is fixed in a fixed frame of reference. V is bounded by the surface S and $\mathbf{n}$ represents a unit normal on S outward from V.

A fluid occupies the space of which V is a subset. The fluid has velocity $\mathbf{u}(\mathbf{x}, t)$ and density $\rho(\mathbf{x}, t)$. Fluid can flow in and out of V, and its density can change (within V).



The mass contained in V is $\int_V \rho(\mathbf{x}, t) dV$.

The flux of mass *into* V ($\mathbf{n}$ points outward) is $-\int_S \rho\mathbf{u}\cdot\mathbf{n} dS$. The quantity $\mathbf{j} = \rho\mathbf{u}$ is called the *mass flux density.*



In time dt the fluid on the surface dS is transported distance $\mathbf{u}\cdot\mathbf{n}$ in a direction perpendicular to dS. Thus, the mass transported is $\rho(\mathbf{u}\cdot\mathbf{n})dtdS$ and the rate of mass transport is this quantity divided by dt.

So the rate of change of mass in V must equal the rate of change of mass in/out of V through S. So

$$\frac{d}{dt}\int_V \rho(\mathbf{x}, t) dV = -\int_S \rho\mathbf{u}\cdot\mathbf{n} dS$$

Since V does not change with t and using the divergence theorem for the RHS we get

$$\int_V \frac{\partial\rho}{\partial t} dV = -\int_V \nabla\cdot(\rho\mathbf{u}) dV$$

or

$$\int_V \left(\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{u})\right) dV = 0$$

This is true for any fixed V, so must have

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0 \tag{9}$$

at every point in the fluid. This is called the mass conservation equation or the continuity equation.

Remark: The structure of this equation is very general, and applies to *any* conservation law, which can be written

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{j} = 0. \tag{10}$$

Here $\mathbf{j}$ is the *flux* of the quantity in question.

1.9 Incompressibility

Defn A fluid is said to be incompressible if the density of each fluid 'particle' is constant, (i.e. $\frac{D\rho}{Dt} = 0$).

From (9) we have

$$0 = \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = \frac{\partial \rho}{\partial t} + \mathbf{u} \cdot \nabla \rho + \rho \nabla \cdot \mathbf{u} = \frac{D\rho}{Dt} + \rho \nabla \cdot \mathbf{u}$$

So ($\rho > 0$) an incompressible fluid satisfies

$$\nabla \cdot \mathbf{u} = 0. \tag{11}$$

No fluid is completely incompressible, but even gases are often sufficiently incompressible for (11) to apply to their motion. Incompressibility is a valid approximation if

- The flow speed is much less than the velocity of sound;
- Timescales are much larger than (sound frequency)$^{-1}$.

Of course this leaves out everything to do with sound waves, which are due entirely to compressible effects.

Before we go on, we will revise more material from vector calculus.

Appendix B goes in here

1.10 Streamfunction for two-dimensional, incompressible flows

In most circumstances, the incompressibility condition (11) is awkward to satisfy. Some progress can be made by using the following theorem:
If $\nabla \cdot \mathbf{u} = 0$, then it follows that there exists a vector field $\mathbf{A}(\mathbf{x}, t)$ s.t.

$$\mathbf{u} = \nabla \times \mathbf{A}.$$

The vector field $\mathbf{A}$ is called the *vector potential.* If $\mathbf{u}$ is represented like that, $\mathbf{u}$ is clearly incompressible. However, $\mathbf{A}$ is far from unique: if the gradient of any scalar function is added to it, the same $\mathbf{u}$ results. Thus to find a unique $\mathbf{A}$, constraints must be applied to it, and we are back to square one! However, the situation is different if the flow is two-dimensional. In that case, $\mathbf{A}$ must point in the direction perpendicular to the plane, so it is determined by a single scalar quantity.

(i) Flow in the x-y plane: Here $\mathbf{A} = \psi(x, y, t)\hat{\mathbf{z}}$, so the flow is determined by a single function ψ. This corresponds exactly to the expected degrees of freedom of the flow: there are two components of the velocity, and one constraint $\nabla \cdot \mathbf{u} = 0$. Then

$$\mathbf{u} = \nabla \times \mathbf{A} = \left(\frac{\partial \psi}{\partial y}, -\frac{\partial \psi}{\partial x}, 0 \right)$$

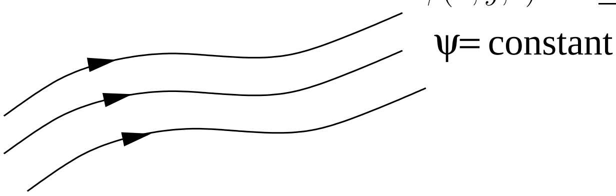
(and clearly $\nabla \cdot \mathbf{u} = 0$).

Reminder: streamlines are given by $\dfrac{dx}{u} = \dfrac{dy}{v}$, where $\mathbf{u} = (u, v, 0)$. So

$$v\mathrm{d}x - u\mathrm{d}y = 0, \qquad \Rightarrow \frac{\partial \psi}{\partial x}\mathrm{d}x + \frac{\partial \psi}{\partial y}\mathrm{d}y = \mathrm{d}\psi = 0$$

(by Chain rule). Therefore $\psi(x, y, t) = \mathit{const}$ on a streamline of the flow.

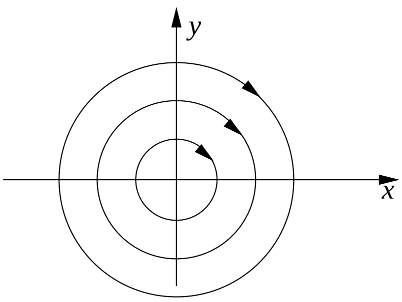
Defn: We call the function $\psi(x, y, t)$ the <u>streamfunction</u> of the flow.



Note: For steady flows, the streamlines do not cross each other and fluid does not cross the streamlines.

Example 1: A <u>vortex</u>. Consider the following flow:

$$\mathbf{u} = \frac{a}{x^2 + y^2}(y, -x)$$



The streamfunction must satisfy

$$\frac{\partial \psi}{\partial y} = \frac{ay}{x^2 + y^2}, \quad \frac{\partial \psi}{\partial x} = \frac{ax}{x^2 + y^2}.$$

From the first equation, $\psi = a \ln(x^2 + y^2)/2 + f(x)$, and then from the second equation $f'(x) = 0$. Thus the streamfunction is

$$\psi = \frac{a}{2} \ln \left(x^2 + y^2 \right),$$

only defined up to a constant, of course. Clearly, the streamlines $\psi = const$ are circles, as expected.

(ii) Polar coordinates: The flow is still in the x-y plane, but the streamfunction is written in polar coorinates: $\mathbf{A} = \psi(r, \theta, t)\hat{\mathbf{z}}$. Then using the formula for the curl in cylindrical polars,

$$\mathbf{u} = \nabla \times \mathbf{A} = \left(\frac{1}{r}\frac{\partial \psi}{\partial \theta}, -\frac{\partial \psi}{\partial r}, 0 \right),$$

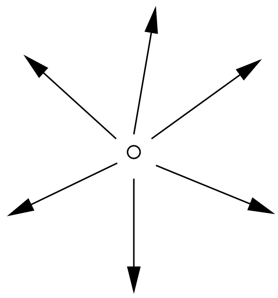
where $\mathbf{u} = (u_r, u_\theta, 0)$. From the geometrical interpretation of ψ it is clear that streamlines are given by $\psi(r, \theta) = const$. (of course this can be checked by direct calculation). Thus

$$\psi(r, \theta, t) = \psi_C(r \cos\theta, r \sin\theta, t),$$

where we have written ψ_C for the streamfunction in Cartesians.

Example 1:

$$\psi = \frac{a}{2} \ln\left(x^2 + y^2\right) = a \ln r.$$



Example 2: A simple <u>source</u>: the flow is purely radial. So $u_\theta = 0$ and u_r is independent of θ.

$$\left.\begin{aligned} u_r &= f(r) = \frac{1}{r}\frac{\partial \psi}{\partial \theta} \\ u_\theta &= 0 = -\frac{\partial \psi}{\partial r} \end{aligned}\right\}$$

We must have $\psi = \psi(\theta)$ but $\partial\psi/\partial\theta$ independent of θ. So $\psi = A\theta$ where A constant. Then $f(r) = A/r$.

Defn: The <u>source strength</u> is the flux of fluid from the source point. The flow is incompressible, so the flux at the origin equals the flux through any closed boundary surrounding the origin. Found, most conveniently, by measuring the flux through a circle radius r centred at the origin. The source strength is

$$m = \oint_{\text{circle}} \mathbf{u} \cdot \mathbf{n} ds = \int_0^{2\pi} \mathbf{u} \cdot \mathbf{n} r d\theta = \int_0^{2\pi} \frac{A}{r} r d\theta = 2\pi A.$$

where $\mathbf{n} = \hat{\mathbf{r}}$ in polars. This means the mass flow through a circle surrounding the origin is independent of the radius of the circle, and equals the source of mass flux at the origin. In fact, one can show that the mass flux through *any* closed surface is m. (how?) In conclusion, the two-dimensional velocity field is

$$\mathbf{u} = \frac{m}{2\pi r}\hat{\mathbf{r}},$$

and the stream function is

$$\psi = \frac{m\theta}{2\pi}.$$

Note: In Cartesians,

$$\psi = \frac{m}{2\pi} \arctan(y/x).$$