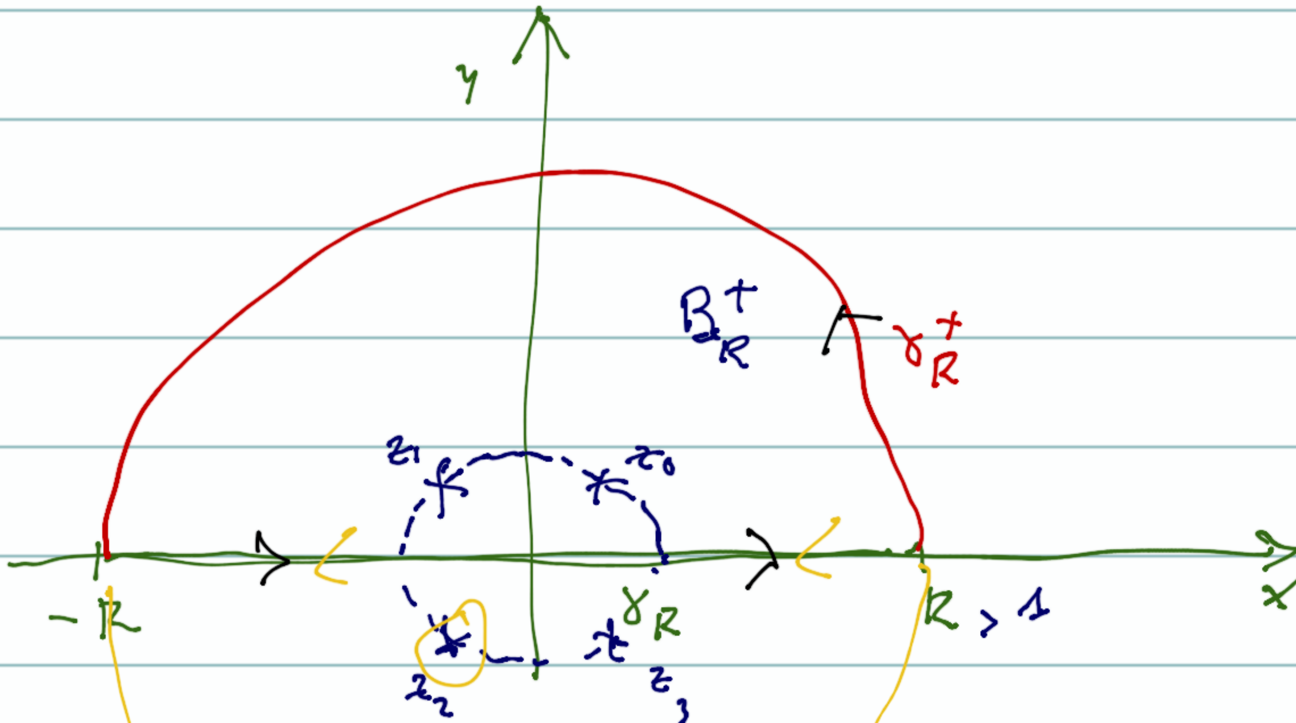
$$\int_{-\infty}^{+\infty} \frac{dx}{1+x^4} = \lim_{R\to+\infty} \int_{-R}^{R} \frac{dx}{1+x^4}$$



$p(z) = 1+z^4$, p si annulla nei punti

$z_0 = e^{i\pi/4}$, $z_1 = e^{i(\frac{3}{4})\pi}$, z_2, z_3 ....

$$B_R^+ = \{ z \in \mathbb{C} \mid |z| < R,\ \operatorname{Im} z > 0 \}$$

$$\int_{\partial B_R^+} \frac{dz}{1+z^4} = 2\pi i \left(\operatorname{Res}\left(\frac{1}{1+z^4}, z_0\right) + \operatorname{Res}\left(\frac{1}{1+z^4}, z_1\right) \right)$$

$$\int_{\partial B_R^+} = \int_{\gamma_R} + \int_{\gamma_R^+} \qquad O\left(\frac{1}{R^3}\right)$$

$$\underline{\int_{\gamma_R} \frac{dz}{1+z^4}} = \underline{\int_{-R}^{R} \frac{dx}{1+x^4}}$$

$$\int_{\gamma_R^+} \underline{\frac{1}{1+z^4}} dz \quad , \quad |z| = R$$

$$|1+z^4| \geq R^4 - 1$$

$$\left| \quad '' \quad \right| \leq L(\gamma_R^+) \frac{1}{R^4-1} = \frac{\pi R}{R^4-1} \to 0$$

$$R \to +\infty$$

$$\lim_{R\to\infty} \int_{-R}^{R} \frac{dx}{1+x^4} = 2\pi i \left(\operatorname{Res}\left(\frac{1}{1+z^4}, z_0\right) + \right.$$

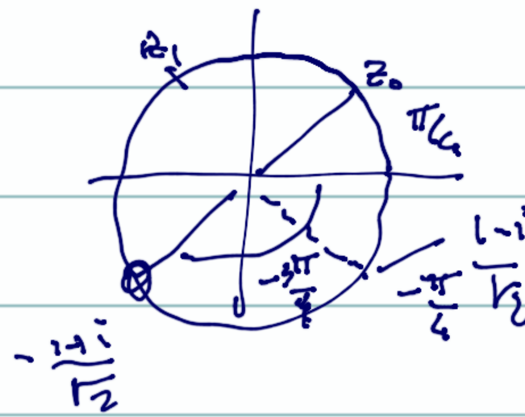
$$\left. + \operatorname{Res}\left(\frac{1}{1+z^4}, z_1\right)\right)$$

$$\mathrm{Res}\left(\frac{1}{1+z^4}, z_0\right) = \frac{1}{4z^3}\Big|_{z=z_0}$$

$$\left(1+z^4 = (z-z_0)(z-z_1)(z-z_2)(z-z_3)\right)$$

$$= \frac{1}{4}\,\frac{1}{z_0^3} = \frac{1}{4}\left(e^{i\pi/4}\right)^{-3} =$$

$$= \frac{1}{4}\,e^{-i3\pi/4}$$



$$\mathrm{Res}\left(\frac{1}{1+z^4}, z_1\right) = \frac{1}{4}\,e^{-3\left(\frac{3}{4}\pi\right)i} =$$

$$= \frac{1}{4}\,e^{-\frac{9}{4}\pi i} \qquad 9 = 8+1$$

$$= \frac{1}{4}\,e^{-i\pi/4}$$

$$\underline{\int_{-\infty}^{+\infty}\frac{dx}{1+x^4}} = \frac{2\pi i}{4}\left(e^{-i\pi/4} + e^{-i\frac{3}{4}\pi}\right) =$$

$$= \frac{2\pi i}{4}\left(\frac{1-i}{\sqrt{2}} - \frac{1+i}{\sqrt{2}}\right) =$$

$$= \frac{\pi i}{2\sqrt{2}}\left(\underline{1}-\underbrace{i}-\underline{1}-\underbrace{i}\right) = \frac{\pi i}{2\sqrt{2}}(-2i) = \frac{\pi}{\sqrt{2}}$$

— . —

Teor. Sia f olomorfa in $\{\operatorname{Im} z \geq 0\}$ con un no. finito di sing isolate

$$z_1, \dots, z_k \in \{\operatorname{Im} z > 0\}.$$

Supp. $|f(z)| \leq C\, |z|^{-1-\alpha}$, per $|z| \geq R$ per opport. costanti

$$C > 0,\ \alpha > 0,\ R > 0.$$

Supp. che f si estende con continuità fino alla retta reale $\mathbb{R} = \{\operatorname{Im} z = 0\}$. Allora

$$\int_{-\infty}^{+\infty} f(x)\,dx = 2\pi i \sum_{k=1}^{K} \operatorname{Res}(f, z_k).$$

$$\int_0^{2\pi} R(\cos\vartheta, \sin\vartheta)\, d\vartheta$$

$R(u,v)$ funz. raz. di u e v

$$\cos\vartheta = \frac{e^{i\vartheta} + e^{-i\vartheta}}{2}, \quad \sin\vartheta = \frac{e^{i\vartheta} - e^{-i\vartheta}}{2i}$$

$$z = e^{i\vartheta}, \quad \cos\vartheta = \frac{z + \frac{1}{z}}{2}, \quad \sin\vartheta = \frac{z - \frac{1}{z}}{2i}$$

$$R(\cos\vartheta, \sin\vartheta) = R\left(\frac{z+z^{-1}}{2}, \frac{z-z^{-1}}{2i}\right) \quad \text{è}$$

una funz. raz. di z.

$$z = e^{i\vartheta}, \quad dz = i e^{i\vartheta}\, d\vartheta$$

$$d\vartheta = \frac{1}{i e^{i\vartheta}}\, dz = \frac{dz}{iz}$$

$$\int_0^{2\pi} R(\cos\vartheta, \sin\vartheta)\, d\vartheta = \boxed{\int_{\partial B_1(0)} R\left(\frac{z+z^{-1}}{2}, \frac{z-z^{-1}}{2i}\right) \frac{dz}{iz}}$$

Esempio

$$\int_0^{2\pi} \frac{d\vartheta}{2+\cos\vartheta} =$$

$$= \int_{\partial B_1(0)} \frac{1}{2+\frac{z+z^{-1}}{2}} \, \frac{dz}{iz} =$$

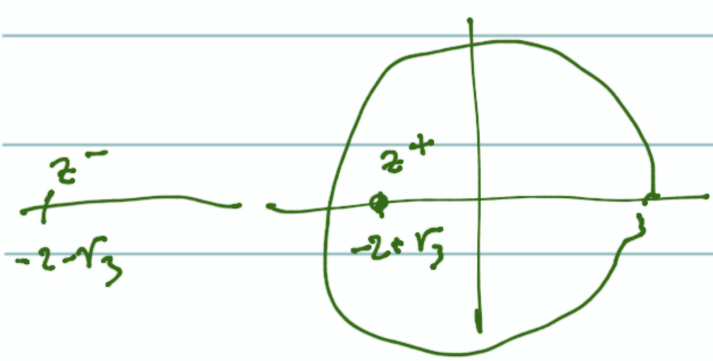
$$= \frac{1}{i} \int_{\partial B_1(0)} \boxed{\frac{dz}{2z+\frac{z^2}{2}+\frac{1}{2}}} =$$

$$= \frac{1}{i} \int_{\partial B_1(0)} \frac{2\,dz}{z^2+4z+1}$$

$$z^2+4z+1 = z^2+4z+4-3 = (z+2)^2$$

ha radici

$$z_\pm = -2 \pm \sqrt{3} \quad \begin{cases} -2+\sqrt{3} \in B_1(0) \\ -2-\sqrt{3} \notin B_1(0) \end{cases}$$



$$\int_0^{2\pi} \frac{d\vartheta}{2+\cos\vartheta} = \frac{1}{\cancel{i}}\, 2\pi \cancel{i}\, \mathrm{Res}\left(\frac{2}{z^2+4z+1}, z^+\right) =$$

$$= 2\pi \left.\frac{2}{2z+4}\right|_{z=z^+} = \frac{4\pi}{2z^+ + 4}$$

$$= \frac{4\pi}{2(-2+\sqrt{3})+4} = \frac{4\pi}{2\sqrt{3}} = \frac{2\pi}{\sqrt{3}}$$

Esercizi

$$\int_0^{2\pi} \frac{d\vartheta}{\sin^2\vartheta + 4\cos^2\vartheta}$$

$$\int_0^{2\pi} \frac{d\vartheta}{a+b\sin\vartheta}, \quad 0<b<a$$

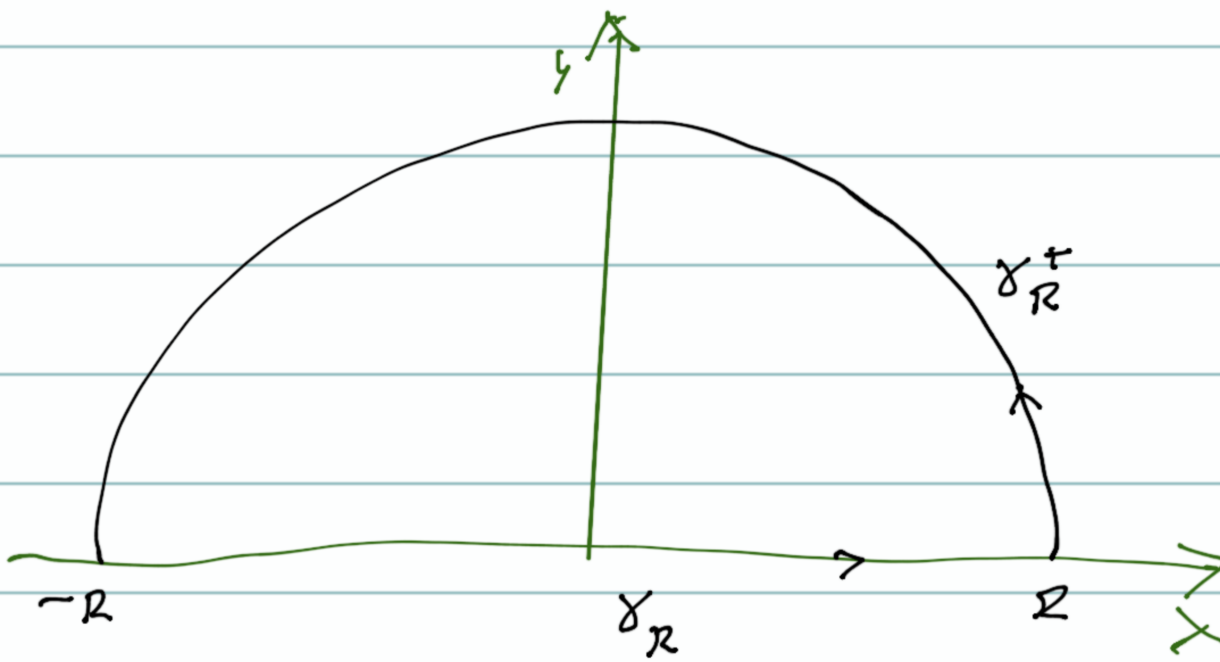
— c —

$$\int_{-\infty}^{+\infty} \frac{\cos x}{1+x^2}\, dx$$

$$\frac{\cos x}{1+x^2} \xrightarrow{?} \frac{\cos(z)}{1+z^2}$$

$$\cos x = \mathrm{Re}\left(e^{ix}\right)$$

$$\int_{-\infty}^{+\infty} \frac{\cos x}{1+x^2}\,dx = \mathrm{Re}\left(\int_{-\infty}^{+\infty} \underbrace{\frac{e^{ix}}{1+x^2}}_{}\,dx\right)$$



$$\int_{\gamma_R^+} \frac{e^{iz}}{1+z^2}\,dz =$$

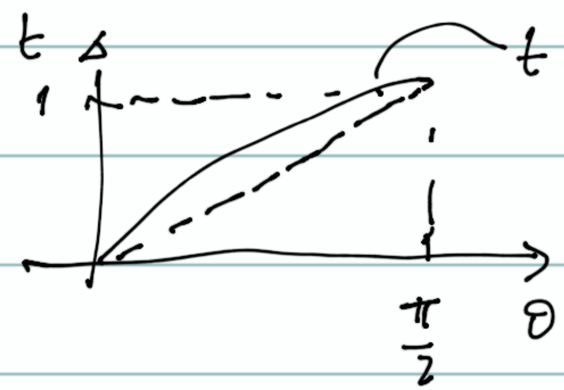
$$z = Re^{i\vartheta}\,,\quad 0<\vartheta<\pi$$

$$= \int_0^{\pi} \frac{1}{1+R^2 e^{i2\vartheta}} \exp\left\{iRe^{i\vartheta}\right\} Rie^{i\vartheta}\,d\vartheta =$$

$$= \int_0^{\pi} \frac{1}{||} \exp\left\{iR\cos\vartheta \underbrace{-R\sin\vartheta}_{}\right\} \underbrace{Rie^{i\vartheta}}_{}\,d\vartheta$$

$$\left|\int_{\gamma_R^+}\right| \le \int_0^{\pi} \frac{1}{R^2-1}\, e^{-R\sin\vartheta}\, R\, d\vartheta =$$

$$= 2\int_0^{\pi/2} \frac{R}{R^2-1}\, e^{-R\sin\vartheta}\, d\vartheta$$



$$\sin\vartheta \ge \frac{2}{\pi}\vartheta$$

$$0 \le \vartheta \le \frac{\pi}{2}$$

$$\left|\int_{\gamma_R^+}\right| \le 2\int_0^{\pi/2} \frac{R}{R^2-1}\, e^{-R\frac{2}{\pi}\vartheta}\, d\vartheta =$$

$$= 2\,\frac{R}{R^2-1}\left(\frac{-1}{R\,\frac{2}{\pi}}\right) e^{-R\frac{2}{\pi}\vartheta}\Big|_0^{\pi/2} =$$

$$= \frac{2R}{R^2-1}\left(\frac{-1}{R\,\frac{2}{\pi}}\right)\left(e^{-R}-1\right) =$$

$$\le \frac{\cancel{2}\cancel{R}}{R^2-1}\,\frac{1}{\cancel{R}\,\frac{\cancel{2}}{\pi}} = \pi\,\frac{1}{R^2-1} \to 0$$

$$R \to \infty$$

$$\underbrace{\lim_{R\to+\infty} \int_{-R}^{R} \frac{\cos x}{1+x^2}\,dx}_{} =$$

$$= \lim_{R\to+\infty} \underbrace{\mathrm{Re}\left(\int_{\partial B_R^+} \frac{e^{iz}}{1+z^2}\,dz \right)}_{} =$$

$$= \mathrm{Re}\left(2\pi i \,\mathrm{Res}\left(\frac{e^{iz}}{1+z^2}, i \right)\right)$$

$$= \mathrm{Re}\left(\cancel{2\pi i} \frac{e^{-1}}{\cancel{2i}} \right) = \frac{\pi}{e}$$

Lemma di Jordan

Sia g continua sull' ins.

$$\{ z \in \mathbb{C} \mid \mathrm{Im}\, z \geq 0,\ |z| \geq R_0 \}$$

Supp.

$$\lim_{R\to+\infty} \underbrace{\max_{\substack{|z|=R \\ \mathrm{Im}\, z \geq 0}} |g(z)|}_{M_R} = 0$$

Allora, $\forall\ \alpha > 0$

$$\lim_{R\to\infty} \int_{\gamma_R^+} g(z)\, e^{i\alpha z}\, dz = 0$$

Dim. Sia $M_R = \max\{|g(z)| \mid |z|=R,\ \operatorname{Im} z \geq 0\}$

Per hp: $M_R \to 0$ per $R \to +\infty$.

$$\left| \int_{\gamma_R^+} g(z)\, e^{i\alpha z}\, dz \right| \leq M_R \int_0^{\pi} e^{-\alpha R \sin\vartheta} R\, d\vartheta =$$

$$= 2M_R \int_0^{\pi/2} e^{-\alpha R \sin\vartheta} R\, d\vartheta$$

$\boxed{\sin\vartheta \geq \frac{2}{\pi}\vartheta,}$ $\quad 0 \leq \vartheta \leq \pi/2$

$$= 2M_R \int_0^{\pi/2} e^{-\alpha R \frac{2}{\pi}\vartheta} R\, d\vartheta =$$

$$= 2M_R \left[-\frac{1}{\alpha \cancel{R} \frac{2}{\pi}} e^{-\alpha R \frac{2}{\pi}\vartheta} \right]_0^{\pi/2} \cancel{R} \leq$$

$$\le \underline{2M_R} \frac{1}{\alpha \frac{2}{\pi}} \longrightarrow 0 \qquad R \to \infty \qquad \square$$

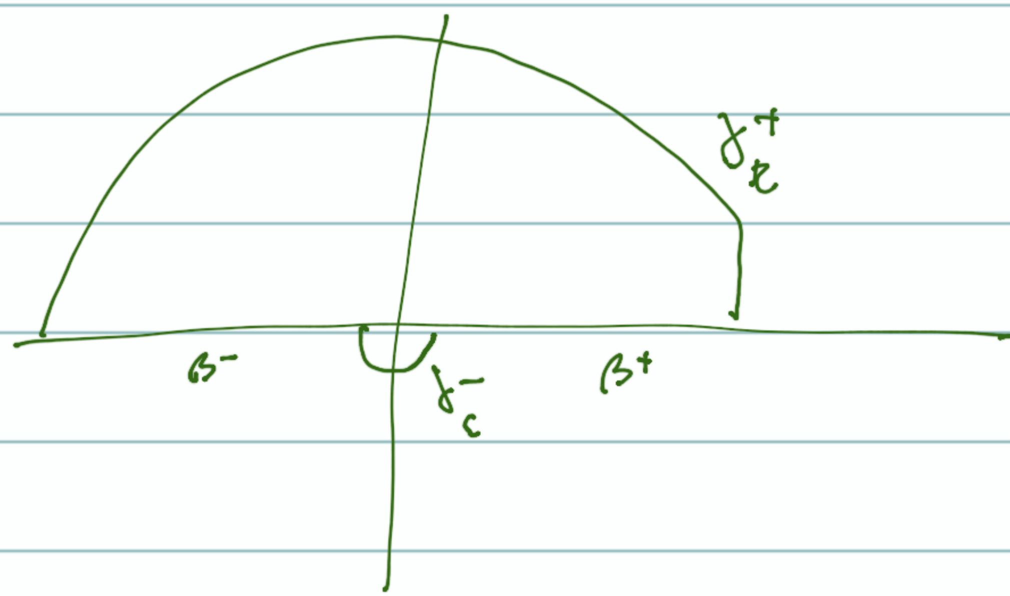
<u>Esempio</u>

$$\lim_{R \to +\infty} \int_{-R}^{R} \frac{x^3}{1+x^4} \sin x \, dx$$

$$\int_0^{+\infty} \frac{\sin x}{x} \, dx =$$

$$= \frac{1}{2} \lim_{R \to +\infty} \int_{-R}^{R} \frac{\sin x}{x} \, dx =$$

$$= \frac{1}{2} \lim_{R \to +\infty} \lim_{\varepsilon \to 0^+} \left(\int_{-R}^{-\varepsilon} + \int_{\varepsilon}^{R} \right) \frac{\sin x}{x} \, dx$$



$$G_{\varepsilon,R} = B_R^+ \cup B_\varepsilon^-$$

$$2\pi i\,\mathrm{Res}\left(\frac{e^{iz}}{z},0\right) = \int_{\partial G_{\varepsilon,R}} \frac{e^{iz}}{z}\,dz = \int_{\beta^-+\beta^+} + \overset{\text{Calcoli}}{\int_{\gamma_\varepsilon^-}} + \overset{\text{Jordan}}{\int_{\gamma_R^+}}$$