Alcune teorie cui ancorare le nostre riflessioni

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Assiomi $\mathcal{G}$ **:** *Chiusura universale* di ciascuna delle formule

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UNDECIDABILITY OF THE ELEMENTARY THEORY OF GROUPS

We consider in this paper an axiomatic theory G with standard formalization, called the *elementary theory of groups* and characterized by the following stipulations: the only non-logical constant of G is the binary operation symbol $\circ$; the set of non-logical axioms of G consists of the three sentences:

Γ_1. $x \circ (y \circ z) = (x \circ y) \circ z$.

Γ_2. $\bigvee z(x = y \circ z)$.

Γ_3. $\bigvee y(x = y \circ z)$.

Hence, a sentence is valid in G if and only if it is satisfied in every system $\langle G, \circ \rangle$ which is a group in the sense of modern algebra (see I.2). The purpose of the paper is to show that the *elementary theory of groups is undecidable.* [1]

Che bastino questi a garantire l'esistenza di un elem. neutro ecc. ?

for mathematics should begin with number theory. The first semi-axiomatic presentation of this subject was given by Dedekind in 1879 and, in a slightly modified form, has come to be known as Peano's postulates.† It can be formulated as follows:

(P1) 0 is a natural number.‡

(P2) If x is a natural number, there is another natural number denoted by x' (and called the *successor* of x).§

(P3) $0 \neq x'$ for every natural number x.

(P4) If $x' = y'$, then $x = y$.

(P5) If Q is a property that may or may not hold for any given natural number, and if (I) 0 has the property Q and (II) whenever a natural number x has the property Q, then x' has the property Q, then all natural numbers have the property Q (mathematical induction principle).

These axioms, together with a certain amount of set theory, can be used to develop not only number theory but also the theory of rational, real and complex numbers (see Mendelson, 1973). However, the axioms involve certain intuitive notions, such as 'property', that prevent this system from being a rigorous formalization. We therefore shall build a first-order theory S that is based upon Peano's postulates and seems to be adequate for the proofs of all the basic results of elementary number theory.

The proper axioms of S are:

(S1) $x_1 = x_2 \Rightarrow (x_1 = x_3 \Rightarrow x_2 = x_3)$
(S2) $x_1 = x_2 \Rightarrow x_1' = x_2'$
(S3) $0 \neq x_1'$
(S4) $x_1' = x_2' \Rightarrow x_1 = x_2$
(S5) $x_1 + 0 = x_1$
(S6) $x_1 + x_2' = (x_1 + x_2)'$
(S7) $x_1 \cdot 0 = 0$
(S8) $x_1 \cdot (x_2)' = (x_1 \cdot x_2) + x_1$
(S9) $\mathscr{B}(0) \Rightarrow ((\forall x)(\mathscr{B}(x) \Rightarrow \mathscr{B}(x')) \Rightarrow (\forall x)\mathscr{B}(x))$ for any wf $\mathscr{B}(x)$ of S.

We shall call (S9) the *principle of mathematical induction*. Notice that axioms (S1)–(S8) are particular wfs, whereas (S9) is an axiom schema providing an infinite number of axioms.†

Axioms (S3) and (S4) correspond to Peano postulates (P3) and (P4), respectively. Peano's axioms (P1) and (P2) are taken care of by the presence of 0 as an individual constant and f_1^1 as a function letter. Our axioms (S1) and (S2) furnish some needed properties of equality; they would have been assumed as intuitively obvious by Dedekind and Peano. Axioms (S5)–(S8) are the recursion equations for addition and multiplication. They were not assumed by Dedekind and Peano because the existence of operations + and · satisfying (S5)–(S8) is derivable by means of intuitive set theory, which was presupposed as a background theory (see Mendelson, 1973, chapter 2, Theorems 3.1 and 5.1).

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$$\langle \mathcal{C}\,,\, \mathcal{F}\,,\, \mathcal{R} \rangle \;=\; \langle \{\, 0 \,\}\,,\, \{ \mathsf{s}_{/1}\,,\, +_{/2}\,,\, *_{/2} \}\,,\, \{ =_{/2}\,,\, \leqslant_{/2} \} \rangle\,.$$

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Assiomi $\mathcal{P}$ **:** Chiusura universale di ciascuna delle formule

$$\begin{aligned}
\mathsf{s}\,x &\neq 0\,, \\
\mathsf{s}\,x = \mathsf{s}\,y &\rightarrow x = y\,, \\
x + 0 &= x\,, \\
x + \mathsf{s}\,y &= \mathsf{s}(x + y)\,, \\
x * 0 &= 0\,, \\
x * \mathsf{s}\,y &= (x * y) + x\,, \\
x \leqslant 0 &\rightarrow x = 0\,, \\
x \leqslant \mathsf{s}\,y &\rightarrow x \leqslant y \vee x = \mathsf{s}\,y\,, \\
x \leqslant y &\vee y \leqslant x\,,
\end{aligned}$$

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Assiomi $\mathcal{P}$ **:** Chiusura universale di ciascuna delle formule

$$\begin{array}{rcl} \mathsf{s}\,x & \neq & 0 , \\ \mathsf{s}\,x = \mathsf{s}\,y & \rightarrow & x = y , \\ x + 0 & = & x , \\ x + \mathsf{s}\,y & = & \mathsf{s}(x + y) , \\ x * 0 & = & 0 , \\ x * \mathsf{s}\,y & = & (x * y) + x , \\ x \leqslant 0 & \rightarrow & x = 0 , \\ x \leqslant \mathsf{s}\,y & \rightarrow & x \leqslant y \ \vee \ x = \mathsf{s}\,y , \\ x \leqslant y & \vee & y \leqslant x , \end{array}$$

nonché *di* ogni esemplare dello *schema d'induzione*

$$\gamma(0) \ \rightarrow \ \forall x \left(\gamma(x) \rightarrow \gamma(\mathsf{s}\,x) \right) \ \rightarrow \ \forall x \, \gamma(x) .$$

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γ, *formula*, passa sul serio in rassegna tutti i sotto-insiemi $\boldsymbol{\gamma}$ di $\mathbb{N}$?

Altra riflex.-lampo sull'induzione

Che vi sia un insieme d'assiomi equivalente a $\mathcal{P}$ ma finito ?

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([Davis(1993), pagg. 40–41])

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Assiomi $\mathcal{W}$ **:** I due enunciati

$$\exists z \,\forall v\; v \notin z\,,$$

$$\forall x \,\forall e \,\exists w \,\forall u \Big(\; u \in w \;\leftrightarrow\; u \in x \;\vee\; u = e \;\Big).$$

Conviene potenziare gli assiomi $\mathcal{W}$?

Aggiungiamo il solito postulato tipico delle teorie degli insiemi *puri*, l'***assioma di estensionalità***

‖ "non ci sono due insiemi con gli stessi elementi":

$$\forall x \forall y \Big(\forall u \,(u \in x \leftrightarrow u \in y) \rightarrow x = y \Big).$$

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