$$\underline{\int_{-\infty}^{+\infty} e^{-x^2 - i\xi x}\, dx} \qquad , \quad \xi \in \mathbb{R}.$$

$$-x^2 - i\xi x = -\left(x + \frac{i\xi}{2}\right)^2 - \frac{\xi^2}{4}$$

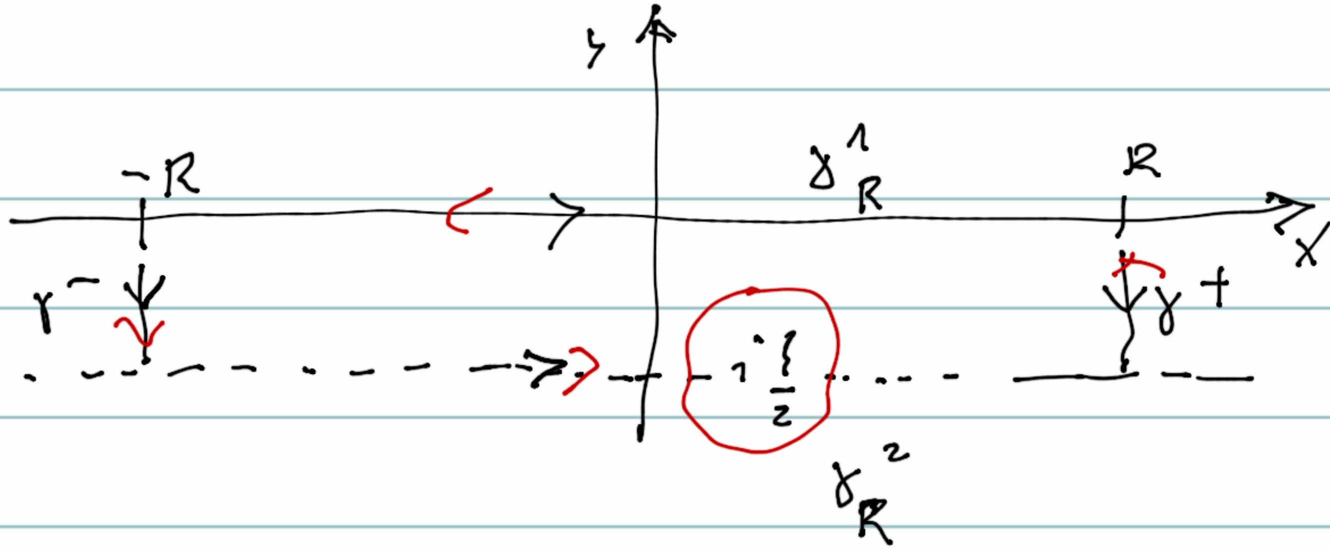
$$\int_{-\infty}^{+\infty} e^{-x^2 - i\xi x}\, dx = e^{-\xi^2/4} \underline{\int_{-\infty}^{+\infty} e^{-\left(x + \frac{i\xi}{2}\right)^2} dx}$$

$$f(z) = e^{-\left(z + \frac{i\xi}{2}\right)^2}.$$

$z + \frac{i\xi}{2} \in \mathbb{R}$ sse $z = x + iy$ è t.c.

$$\boxed{y = -\frac{\xi}{2}}$$

Supp. per il momento $\xi > 0$



$$\int f(z)\,dz = 0$$

$$-\gamma_R^1 + \gamma^- + \gamma_R^2 - \gamma^+$$

$$\int_{\gamma_R^1} f(z)\,dz = \int_{-R}^{R} f(x)\,dx$$

$$\int_{\gamma_R^1} = \int_{\gamma_R^2} + \int_{\gamma^-} - \int_{\gamma^+}$$

$$\int_{\gamma^-} f(z)\,dz = \int_0^{-\frac{\xi}{2}} f(-R+iy)\,dy =$$

$$\left(f(z) = e^{-\left(z+\frac{i\xi}{2}\right)^2} \right)$$

$$= \int_0^{-\frac{\xi}{2}} \exp\Big\{-\big(-R + \underbrace{iy + i\tfrac{\xi}{2}}_{O(1)}\big)^2\Big\}\, dy \qquad R \to \infty$$

$$\sim O(e^{-R^2}) \longrightarrow 0 \qquad R \to +\infty.$$

Idem : $\int_{\gamma^+} \longrightarrow 0$ pr $R \to +\infty$.

$$\int_{\gamma_R^2} = \int_{-R}^{R} e^{-(x - i\frac{\xi}{2} + i\frac{\xi}{2})^2}\, dx = \int_{-R}^{R} e^{-x^2} dx \Rightarrow$$

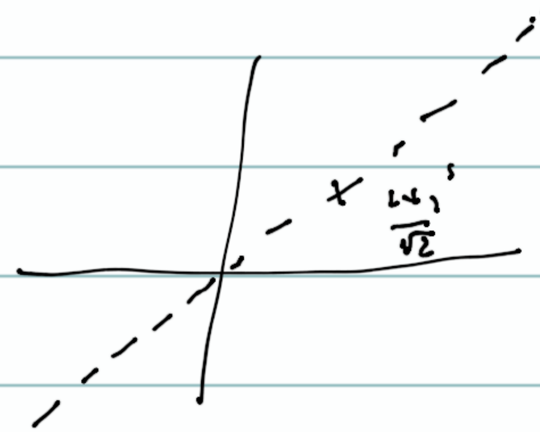
$$\longrightarrow \int_{-\infty}^{+\infty} e^{-x^2} dx = \sqrt{\pi}$$

pr $\xi > 0$: $\displaystyle\int_{-\infty}^{+\infty} e^{-x^2 - i\xi x}\, dx = \sqrt{\pi}\, e^{-\xi^2/4}$

Analog pr $\xi < 0$.

$$\lim_{R\to\infty} \int_{-R}^{R} e^{ix^2}\, dx$$

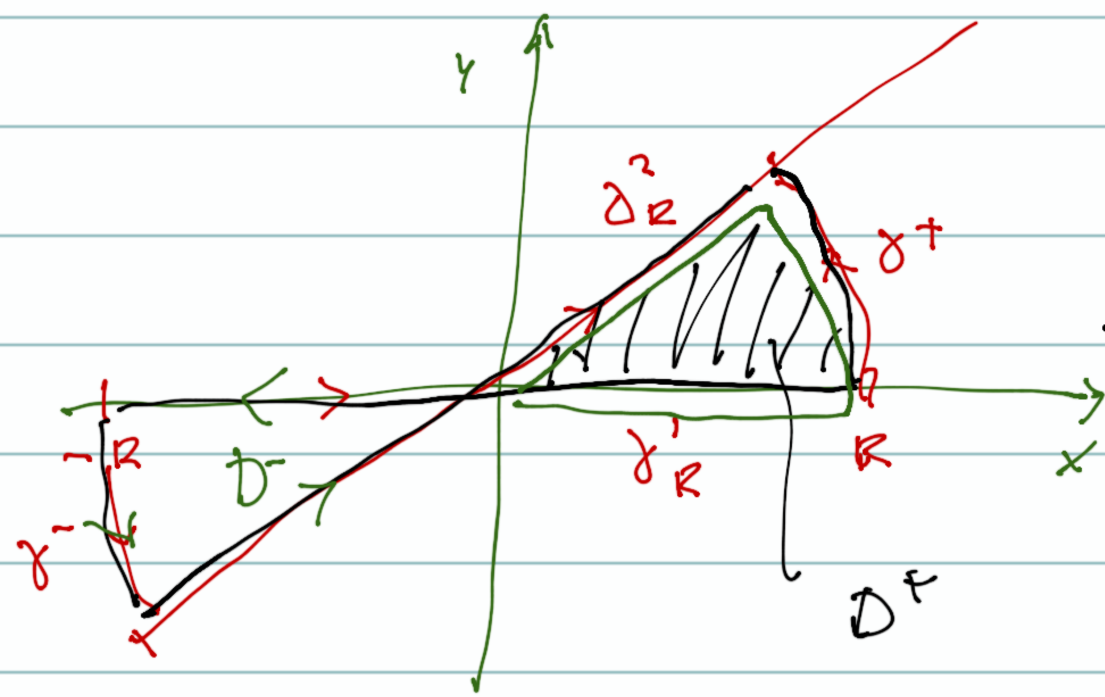
$$f(z) = e^{iz^2}$$



so $\quad z = \frac{1+i}{\sqrt{2}}\, t$

$$z^2 = i t^2$$

$$f(z) = e^{-t^2}$$



$$\int_{\gamma_R^1 + \gamma^+ - \gamma_R^2 + \gamma^-} f(z)\, dz = \int_{\partial D^+} - \int_{\partial D^-} = 0$$

$$\int_{\gamma^+} e^{iz^2}\, dz = \int_0^{\pi/4} e^{i(Re^{i\vartheta})^2}\, \underline{iRe^{i\vartheta}}\, d\vartheta$$

$$\{ z = Re^{i\vartheta} \}$$

$$(Re^{i\vartheta})^2 = R^2(\cos 2\vartheta + i \sin 2\vartheta)$$

$$i(Re^{i\vartheta})^2 = \underline{-R^2 \sin 2\vartheta} + \underline{i R^2 \cos 2\vartheta}$$

$$\left| \int_{\gamma^+} \right| \le \int_0^{\pi/4} e^{-R^2 \sin 2\vartheta} R\, d\vartheta$$

$$0 \le \vartheta \le \frac{\pi}{4}, \quad 0 \le 2\vartheta \le \frac{\pi}{2}$$

$$\sin 2\vartheta \ge \frac{4\vartheta}{\pi}$$

$$\left| \int_{\gamma^+} \right| \leq \int_0^{\pi/4} e^{-R^2 \frac{4\vartheta}{\pi}} R \, d\vartheta \underset{R\to\infty}{\longrightarrow} 0$$

Analog: $\int_{\gamma^-} \longrightarrow 0$ für $R \to \infty$.

$$\int_{\gamma_R^2} f(z)\, dz = \int_{-R}^{R} e^{i\left(\frac{1+i}{\sqrt{2}} t\right)^2} dt =$$

$$= \int_{-R}^{R} e^{-t^2} dt \underset{R\to+\infty}{\longrightarrow} \sqrt{\pi}.$$

$$\lim_{R\to+\infty} \int_{-R}^{R} \frac{x^2 e^{-ix}}{x^3 - i}\, dx$$

$$\frac{x^2}{x^3 - i} \longrightarrow \underbrace{\frac{z^2}{z^3 - i}}_{f(z)} \sim \frac{1}{|z|} \quad \text{per} \quad |z| \to \infty$$

$z^3 - i = 0$ se $z = e^{i\vartheta}$ e

$$e^{i3\vartheta} = i = e^{i\pi/2}$$

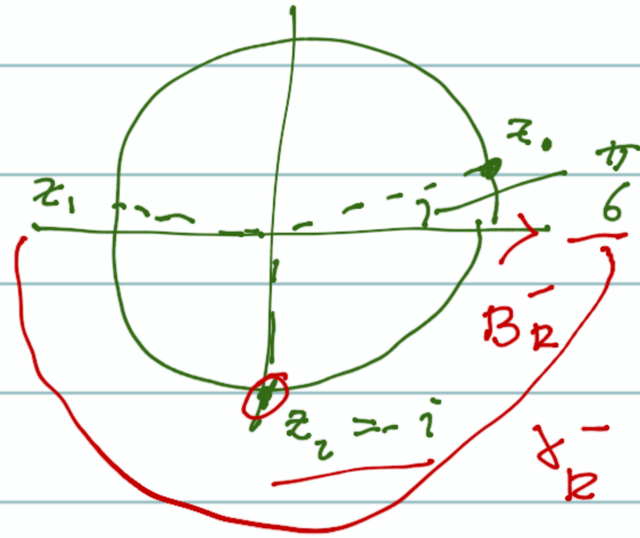
$$3\vartheta = \frac{\pi}{2} + 2k\pi \qquad k \in \mathbb{Z}$$

$$\vartheta_k = \frac{\pi}{6} + \frac{2k}{3}\pi \quad , \quad k = 0, 1, 2$$

$$\vartheta_0 = \frac{\pi}{6}, \quad \vartheta_1 = \frac{5}{6}\pi, \quad \vartheta_2 = \frac{3}{2}\pi$$

$$\frac{1}{6} + \frac{2}{3} = \frac{1+4}{6} = \frac{5}{6}$$

...



$$\int_{-R}^{R} \frac{x^2 e^{-ix}}{x^3 - i}\, dx = -\int_{\partial B_R^-} \frac{z^2 e^{-iz}}{z^3 - i}\, dz + \int_{\gamma_R^-} \cdots$$

Lemma 1. Jordan :

$$\lim_{R\to\infty} \int_{\gamma_R^-} \cdots = 0 .$$

$$\to \int_{\partial B_R^-} \frac{z^2 e^{-iz}}{(z^3 - i)}\, dz = 2\pi i \operatorname{Res}(\cdots, -i)$$

$$= 2\pi i \frac{\cancel{(-i)^2} e^{-1}}{3\cancel{(-i)^2}} = \frac{2\pi i}{3e}$$

$$\lim_{R\to\infty}\int_{-R}^{R} \frac{x^2 e^{-ix}}{x^3-i}\,dx = -\frac{2\pi i}{3e}$$

$$\frac{x^2}{x^3-i} = \frac{x^2(x^3+i)}{x^6+1} =$$

$$= \frac{x^5}{x^6+1} + i\,\frac{x^2}{x^6+1}$$

$$\frac{x^2}{x^3-i}\,e^{-ix} = \left(\frac{x^5}{x^6+1} + i\,\frac{x^2}{x^6+1}\right)(\cos x - i\sin x)$$

$$= \underbrace{\left[\frac{x^5}{x^6+1}\cos x + \frac{x^2}{x^6+1}\sin x\right]}_{\text{dispari}} +$$

$$+ i\underbrace{\left[\frac{x^5}{x^6+1}\sin x + \frac{x^2}{x^6+1}\cos x\right]}_{\text{simm. pari}}$$