## 9. Regular and rational maps.

In the following $K$ is an algebraically closed field.

### a) Regular maps.

Let $X, Y$ be quasi–projective varieties (or more generally locally closed sets). Let $\phi : X \to Y$ be a map.

**9.1. Definition.** $\phi$ is a *regular map* or a *morphism* if
(i) $\phi$ is continuous;
(ii) $\phi$ preserves regular functions, i.e. for all $U \subset Y$ ($U$ open and non–empty) and for all $f \in \mathcal{O}(U)$, then $f \circ \phi \in \mathcal{O}(\phi^{-1}(U))$:

$$\begin{array}{ccccc} X & \stackrel{\phi}{\longrightarrow} & Y & & \\ \uparrow & & \uparrow & & \\ \phi^{-1}(U) & \stackrel{\phi|}{\longrightarrow} & U & \stackrel{f}{\to} & K \end{array}$$

Note that:
a) for all $X$ the identity map $1_X : X \to X$ is regular;
b) for all $X$, $Y$, $Z$ and regular maps $X \stackrel{\phi}{\to} Y$, $Y \stackrel{\psi}{\to} Z$, the composite map $\psi \circ \phi$ is regular.

An *isomorphism* of varieties is a regular map which possesses regular inverse, i.e. a regular $\phi : X \to Y$ such that there exists a regular $\psi : Y \to X$ verifying the conditions $\psi \circ \phi = 1_X$ and $\phi \circ \psi = 1_Y$. In this case $X$ and $Y$ are said to be isomorphic, and we write: $X \simeq Y$.

If $\phi : X \to Y$ is regular, there is a natural $K$–homomorphism $\phi^* : \mathcal{O}(Y) \to \mathcal{O}(X)$, called the *comorphism associated to* $\phi$, defined by: $f \to \phi^*(f) := f \circ \phi$.

The construction of the comorphism is functorial, which means that:
a) $1_X^* = 1_{\mathcal{O}(X)}$;
b) $(\psi \circ \phi)^* = \phi^* \circ \psi^*$.

This implies that, if $X \simeq Y$, then $\mathcal{O}(X) \simeq \mathcal{O}(Y)$. In fact, if $\phi : X \to Y$ is an isomorphism and $\psi$ is its inverse, then $\phi \circ \psi = 1_Y$, so $(\phi \circ \psi)^* = \psi^* \circ \phi^* = (1_Y)^* = 1_{\mathcal{O}(Y)}$ and similarly $\psi \circ \phi = 1_X$ implies $\phi^* \circ \psi^* = 1_{\mathcal{O}(X)}$.

**9.2. Examples**.

1) The homeomorphism $\phi_i : U_i \to \mathbb{A}^n$ of Proposition 3.2 is an isomorphism.

2) There exist homeomorphisms which are not isomorphisms. Let $Y = V(x^3 - y^2) \subset \mathbb{A}^2$. We have seen (see Exercise 7.2) that $K[X] \not\simeq K[\mathbb{A}^1]$, hence $Y$ is not isomorphic to the affine line. Nevertheless, the following map is regular, bijective and also a homeomorphism (see Exercise 7.1):
$\phi : \mathbb{A}^1 \to Y$ such that $t \to (t^2, t^3)$;

$\phi^{-1} : Y \to \mathbb{A}^1$ is defined by $(x, y) \to \begin{cases} \frac{y}{x} \text{ if } x \neq 0 \\ 0 \text{ if } (x, y) = (0, 0). \end{cases}$

Note that $\phi^{-1}$ is not regular at the point $(0, 0)$.

**9.3. Proposition.** *Let $\phi : X \to Y \subset \mathbb{A}^n$ be a map. Then $\phi$ is regular if and only if $\phi_i := t_i \circ \phi$ is a regular function on $X$, for all $i = 1, \ldots, n$, where $t_1, \ldots, t_n$ are the coordinate functions on $Y$.*

*Proof.* If $\phi$ is regular, then $\phi_i = \phi^*(t_i)$ is regular by definition.

Conversely, assume that $\phi_i$ is a regular function on $X$ for all $i$. Let $Z \subset Y$ be a closed subset and we have to prove that $\phi^{-1}(Z)$ is closed in $X$. Since any closed subset of $\mathbb{A}^n$ is an intersection of hypersurfaces, it is enough to consider $\phi^{-1}(Y \cap V(F)$ with $F \in K[x_1, \ldots, x_n]$:

$$\phi^{-1}(V(F) \cap Y) = \{P \in X | F(\phi(P)) = F(\phi_1, \ldots, \phi_n)(P) = 0\} = V(F(\phi_1, \ldots, \phi_n)).$$

But note that $F(\phi_1, \ldots, \phi_n) \in \mathcal{O}(X)$: it is the composition of $F$ with the regular functions $\phi_1, \ldots, \phi_n$. Hence $\phi^{-1}(V(F) \cap Y)$ is closed, so we can conclude that $\phi$ is continuous. If $U \subset Y$ and $f \in \mathcal{O}(U)$, for any point $P$ of $U$ choose an open neighbourhood $U_P$ such that $f = F_P/G_P$ on $U_P$.

So $f \circ \phi = F_P(\phi_1, \ldots, \phi_n)/G_P(\phi_1, \ldots, \phi_n)$ on $\phi^{-1}(U_P)$, hence it is regular on each $\phi^{-1}(U_P)$ and by consequence on $\phi^{-1}(U)$.

□

If $\phi : X \to Y$ is a regular map and $Y \subset \mathbb{A}^n$, by Proposition 9.2. we can represent $\phi$ in the form $\phi = (\phi_1, \ldots, \phi_n)$, where $\phi_1, \ldots, \phi_n \in \mathcal{O}(X)$ and $\phi_i = \phi^*(t_i)$. $\phi_1, \ldots, \phi_n$ are not arbitrary in $\mathcal{O}(X)$ but such that Im $\phi \subset Y$. If $Y$ is closed in $\mathbb{A}^n$, let us recall that $t_1, \ldots, t_n$ generate $\mathcal{O}(Y)$, hence $\phi_1, \ldots, \phi_n$ generate $\phi^*(\mathcal{O}(Y))$ as $K$-algebra. This observation is the key for the following important result.

**9.4. Theorem.** *Let $X$ be a locally closed algebraic set and $Y$ be an affine algebraic set. Let $Hom(X, Y)$ denote the set of regular maps from $X$ to $Y$ and $Hom(\mathcal{O}(Y), \mathcal{O}(X))$ denote the set of $K$– homomorphisms from $\mathcal{O}(Y)$ to $\mathcal{O}(X)$.*

*Then the map $Hom(X, Y) \to Hom(\mathcal{O}(Y), \mathcal{O}(X))$, such that $\phi : X \to Y$ goes to $\phi^* : \mathcal{O}(Y) \to \mathcal{O}(X)$, is bijective.*

*Proof.* Let $Y \subset \mathbb{A}^n$ and let $t_1, \ldots, t_n$ be the coordinate functions on $Y$, so $\mathcal{O}(Y) = K[t_1, \ldots, t_n]$. Let $u : \mathcal{O}(Y) \to \mathcal{O}(X)$ be a $K$–homomorphism: we want to define a morphism $u^\sharp : X \to Y$ whose associated comorphism is $u$. By the remark above, if $u^\sharp$ exists, its components have to be $u(t_1), \ldots, u(t_n)$. So we define

$$\begin{array}{cccc} u^\sharp : & X & \to & \mathbb{A}^n \\ & P & \to & (u(t_1)(P)), \ldots, u(t_n)(P)). \end{array}$$

This is a morphism by Proposition 9.3. We claim that $u^\sharp(X) \subset Y$. Let $F \in I(Y)$ and $P \in X$: then

$$\begin{aligned}(F(u^\sharp(P)) &= F(u(t_1)(P), \dots, u(t_n)(P)) = \\ &= F(u(t_1), \dots, u(t_n))(P) = \\ &= u(F((t_1, \dots, t_n))(P) \text{ because } u \text{ is } K\text{-homomorphism} = \\ &= u(0)(P) = \\ &= 0(P) = 0.\end{aligned}$$

So $u^\sharp$ is a regular map from $X$ to $Y$.

We consider now $(u^\sharp)^* : \mathcal{O}(Y) \to \mathcal{O}(X)$: it takes a function $f$ to $f \circ u^\sharp = f(u(t_1), \dots, u(t_n)) = u(f)$, so $(u^\sharp)^* = u$. Conversely, if $\phi : X \to Y$ is regular, then $(\phi^*)^\sharp$ takes $P$ to $(\phi^*(t_1)(P), \dots, \phi^*(t_n)(P)) = (\phi_1(P), \dots, \phi_n(P))$, so $(\phi^*)^\sharp = \phi$.

□

Note that, by definition, $1^\sharp_{\mathcal{O}(X)} = 1_X$, for all affine $X$; moreover $(v \circ u)^\sharp = u^\sharp \circ v^\sharp$ for all $u : \mathcal{O}(Z) \to \mathcal{O}(Y)$, $v : \mathcal{O}(Y) \to \mathcal{O}(X)$, $K$–homomorphisms of affine algebraic sets: this means that also this construction is functorial.

The previous results can be rephrased using the language of categories. We introduce a category $\mathcal{C}$ whose objects are the affine algebraic sets over a fixed algebraically closed field $K$ and the morphisms are the regular maps. We consider also a second category $\mathcal{C}'$ with objects the $K$-algebras and morphisms the $K$-homomorphisms. Then there is a contravariant functor that operates on the objects sending $X$ to $\mathcal{O}(X) = K[X]$, and on the morphisms sending $\phi$ to the associated comorphisms $\phi^*$.

If we restrict the class of objects of $\mathcal{C}'$ taking only the finitely generated reduced $K$-algebras (a full subcategory of the previous one), then this functor becomes an equivalence of categories. Indeed the construction of the comorphism establishes a bijection between the Hom sets $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ and $\mathrm{Hom}_{\mathcal{C}'}(\mathcal{O}(Y), \mathcal{O}(X))$. Moreover, for any finitely generated $K$-algebra $A$, there exists an affine algebraic set $X$ such that $A$ is $K$-isomorphic to $\mathcal{O}(X)$. To see this, we choose a finite set of generators of $A$, such that $A = K[\xi_1, \dots, \xi_n]$. Then we can consider the surjective $K$-homomorphism $\Psi$ from the polynomial ring $K[x_1, \dots, x_n]$ to $A$ sending $x_i$ to $\xi_i$ for any $i$. In view of the fundamental theorem of homomorphism, it follows that $A \simeq K[x_1, \dots, x_n]/\ker \Psi$. The assumption that $A$ is reduced then implies that $X := V(\ker \Psi) \subset \mathbb{A}^n$ is an affine algebraic set with $I(X) = \ker \Psi$ and $A \simeq \mathcal{O}(X)$.

We note that changing system of generators for $A$ changes the homomorphism $\Psi$, and by consequence also the algebraic set $X$, up to isomorphism. For instance let $A$ be a polynomial ring in one variable $t$: if we choose only $t$ as system of generators, we get $X = \mathbb{A}^1$, but if we choose $t, t^2, t^3$ we get the affine skew cubic in $\mathbb{A}^3$.

As a consequence of the previous discussion we have the following:

**9.5. Corollary.** *Let $X$, $Y$ be affine algebraic sets. Then $X \simeq Y$ if and only if $\mathcal{O}(X) \simeq \mathcal{O}(Y)$.* □

If $X$ and $Y$ are quasi–projective varieties and $\phi : X \to Y$ is regular, it is not always possible to define a comorphism $K(Y) \to K(X)$. If $f$ is a rational function on $Y$ with $\mathrm{dom} f = U$, it can happen that $\phi(X) \cap \mathrm{dom} f = \emptyset$, in which case $f \circ \phi$ does not exist. Nevertheless, if we assume that $\phi$ is dominant, i.e. $\overline{\phi(X)} = Y$, then certainly $\phi(X) \cap U \neq \emptyset$, hence $\langle \phi^{-1}(U), f \circ \phi \rangle \in K(X)$. We obtain a $K$–homomorphism, which is necessarily injective, $K(Y) \to K(X)$, also denoted by $\phi^*$. Note that in this case, we have: $\dim X \geq \dim Y$. As above, it is possible to check that, if $X \simeq Y$, then $K(X) \simeq K(Y)$, hence $\dim X = \dim Y$. Moreover, if $P \in X$ and $Q = \phi(P)$, then $\phi^*$ induces a map $\mathcal{O}_{Q,Y} \to \mathcal{O}_{P,X}$, such that $\phi^* \mathcal{M}_{Q,Y} \subset \mathcal{M}_{P,X}$. Also in this case, if $\phi$ is an isomorphism, then $\mathcal{O}_{Q,Y} \simeq \mathcal{O}_{P,X}$.

We will see now how to express in practice a regular map when the target is contained in a projective space. Let $X \subset \mathbb{P}^n$ be a quasi–projective variety and $\phi : X \to \mathbb{P}^m$ be a map.

**9.6. Proposition.** *$\phi$ is a morphism if and only if, for any $P \in X$, there exist an open neighbourhood $U_P$ of $P$ and $n+1$ homogeneous polynomials $F_0, \dots, F_m$ of the same degree, in $K[x_0, x_1, \dots, x_n]$, such that, if $Q \in U_P$, then $\phi(Q) = [F_0(Q), \dots, F_m(Q)]$. In particular, for any $Q \in U_P$, there exists an index $i$ such that $F_i(Q) \neq 0$.*

*Proof.* "$\Rightarrow$" Let $P \in X$, $Q = \phi(P)$ and assume that $Q \in U_0$. Then $U := \phi^{-1}(U_0)$ is an open neighbourhood of $P$ and we can consider the restriction $\phi|_U : U \to U_0$, which is regular. Possibly after restricting $U$, using non–homogeneous coordinates on $U_0$, we can assume that $\phi|_U = (F_1/G_1, \dots, F_m/G_m)$, where $(F_1, G_1)$, …, $(F_m, G_m)$ are pairs of homogeneous polynomials of the same degree such that $V_P(G_i) \cap U = \emptyset$ for all index $i$. We can reduce the fractions $F_i/G_i$ to a common denominator $F_0$, so that $\deg F_0 = \deg F_1 = \ldots = \deg F_m$ and $\phi|_U = (F_1/F_0, \dots, F_m/F_0) = [F_0, F_1, \dots, F_m]$, with $F_0(Q) \neq 0$ for $Q \in U$.

"$\Leftarrow$" Possibly after restricting $U_P$, we can assume $F_i(Q) \neq 0$ for all $Q \in U_P$ and suitable $i$. Let $i = 0$: then $\phi|_{U_P} : U_P \to U_0$ operates as follows: $\phi|_{U_P}(Q) = (F_1(Q)/F_0(Q), \dots, F_m(Q)/F_0(Q))$, so it is a morphism by Proposition 9.3. From this remark, one deduces that also $\phi$ is a morphism. □

**9.7. Examples.**

1. Let $X \subset \mathbb{P}^2$, $X = V_P(x_1^2 + x_2^2 - x_0^2)$, the projective closure of the unitary circle. We define $\phi : X \to \mathbb{P}^1$ by

$$[x_0, x_1, x_2] \to \begin{cases} [x_0 - x_2, x_1] \text{ if } (x_0 - x_2, x_1) \neq (0,0); \\ [x_1, x_0 + x_2] \text{ if } (x_1, x_0 + x_2) \neq (0,0). \end{cases}$$

$\phi$ is well–defined because on $X$ $x_1^2 = (x_0 - x_2)(x_0 + x_2)$. Moreover

$$(x_1, x_0 - x_2) \neq (0,0) \Leftrightarrow [x_0, x_1, x_2] \in X \setminus \{[1,0,1]\},$$

$$(x_0 + x_2, x_1) \neq (0,0) \Leftrightarrow [x_0, x_1, x_2] \in X \setminus \{[1,0,-1]\}.$$

The map $\phi$ is the natural extension of the rational function $f : X \setminus \{[1,0,1]\} \to K$ such that $[x_0, x_1, x_2] \to x_1/(x_0 - x_2)$ (Example 8.9, 2). Now the point $P[1,0,1]$, the centre of the stereographic projection, goes to the point at infinity of the line $V_P(x_2)$.

By geometric reasons $\phi$ is invertible and $\phi^{-1} : \mathbb{P}^1 \to X$ takes $[\lambda, \mu]$ to $[\lambda^2 + \mu^2, 2\lambda\mu, \lambda^2 - \mu^2]$ (note the connection with the Pitagorean triples!).

Indeed: the line through $P$ and $[\lambda, \mu, 0]$ has equation: $\mu x_0 - \lambda x_1 - \mu x_2 = 0$. Its intersections with $X$ are represented by the system:

$$\begin{cases} \mu x_0 - \lambda x_1 - \mu x_2 = 0 \\ x_1^2 + x_2^2 - x_0^2 = 0 \end{cases}$$

Assuming $\mu \neq 0$ this system is equivalent to the following:

$$\begin{cases} \mu x_0 - \lambda x_1 - \mu x_2 = 0 \\ \mu^2 x_0^2 = \mu^2 (x_1^2 + x_2^2) = (\lambda x_1 + \mu x_2)^2. \end{cases}$$

Therefore, either $x_1 = 0$ and $x_0 = x_2$, or $\begin{cases} (\mu^2 - \lambda^2) x_1 - 2\lambda\mu x_2 = 0 \\ \mu x_0 = \lambda x_1 + \mu x_2 \end{cases}$, which gives the required expression.

2. *Affine transformations.*

Let $A = (a_{ij})$ be a $n \times n$–matrix with entries in $K$, let $B = (b_1, \dots, b_n) \in \mathbb{A}^n$ be a point. The map $\tau_A : \mathbb{A}^n \to \mathbb{A}^n$ defined by $(x_1, \dots, x_n) \to (y_1, \dots, y_n)$, such that

$$\{y_i = \sum_j a_{ij} x_j + b_i, i = 1, \dots, n,$$

is a regular map called an affine transformation of $\mathbb{A}^n$. In matrix notation $\tau_A$ is $Y = AX + B$. If $A$ is of rank $n$, then $\tau_A$ is said non–degenerate and is an isomorphism: the inverse map $\tau_A^{-1}$ is represented by $X = A^{-1}Y - A^{-1}B$. More in general, an affine transformation from $\mathbb{A}^n$ to $\mathbb{A}^m$ is a map represented in matrix form by $Y = AX + B$, where $A$ is a $m \times n$ matrix and $B \in \mathbb{A}^m$. It is injective if and only if $\mathrm{rk}A = n$ and surjective if and only if $\mathrm{rk}A = m$.

The isomorphisms of an algebraic set $X$ in itself are called automorphisms of $X$: they form a group for the usual composition of maps, denoted *Aut* $X$. If $X = \mathbb{A}^n$, the non–degenerate affine transformations form a subgroup of *Aut* $\mathbb{A}^n$.

If $n = 1$ and the characteristic of $K$ is 0, then *Aut* $\mathbb{A}^1$ coincides with this subgroup. In fact, let $\phi : \mathbb{A}^1 \to \mathbb{A}^1$ be an automorphism: it is represented by a polynomial $F(x)$ such that there exists $G(x)$ satisfying the condition $G(F(t)) = t$

for all $t \in \mathbb{A}^1$, i.e. $G(F(x)) = x$ in the polynomial ring $K[x]$. Then, taking derivatives, we get $G'(F(x))F'(x) = 1$, which implies $F'(t) \neq 0$ for all $t \in K$, so $F'(x)$ is a non–zero constant. Hence, $F$ is linear and $G$ is linear too.

If $n \geq 2$, then $Aut\ \mathbb{A}^n$ is not completely described. There exist non–linear automorphisms of degree $d$, for all $d$. For example, for $n = 2$: let $\phi : \mathbb{A}^2 \to \mathbb{A}^2$ be given by $(x, y) \to (x, y + P(x)$, where $P$ is any polynomial of $K[x]$. Then $\phi^{-1} : (x', y') \to (x', y' - P(x'))$. A very important open problem is the Jacobian conjecture, stating that, in characteristic zero, a regular map $\phi : \mathbb{A}^n \to \mathbb{A}^n$ is an automorphism if and only if the Jacobian determinant $\mid J(\phi) \mid$ is a non-zero constant.

3. *Projective transformations.*

Let $A$ be a $(n+1)\times(n+1)$–matrix with entries in $K$. Let $P[x_0, \dots, x_n] \in \mathbb{P}^n$: then $[a_{00}x_0 + \dots + a_{0n}x_n, \dots, a_{n0}x_0 + \dots + a_{nn}x_n]$ is a point of $\mathbb{P}^n$ if and only if it is different from $[0, \dots, 0]$. So $A$ defines a regular map $\tau : \mathbb{P}^n \to \mathbb{P}^n$ if and only if $\mathrm{rk}A = n+1$. If $\mathrm{rk}A = r < n+1$, then $A$ defines a regular map whose domain is the quasi–projective variety $\mathbb{P}^n \setminus \mathbb{P}(kerA)$. If $\mathrm{rk}A = n + 1$, then $\tau$ is an isomorphism, called a projective transformation. Note that the matrices $\lambda A$, $\lambda \in K^*$, all define the same projective transformation. So $PGL(n + 1, K) := GL(n + 1, K)/K^*$ acts on $\mathbb{P}^n$ as the group of projective transformations.

If $X, Y \subset \mathbb{P}^n$, they are called projectively equivalent if there exists a projective transformation $\tau : \mathbb{P}^n \to \mathbb{P}^n$ such that $\tau(X) = Y$.

**9.8. Theorem.** *Fundamental theorem on projective transformations.*

*Let two $(n+2)$–tuples of points of $\mathbb{P}^n$ in general position be fixed: $P_0, \dots, P_{n+1}$ and $Q_0, \dots, Q_{n+1}$. Then there exists one isomorphic projective transformation $\tau$ of $\mathbb{P}^n$ in itself, such that $\tau(P_i) = Q_i$ for all index $i$.*

*Proof.* Put $P_i = [v_i]$, $Q_i = [w_i]$, $i = 0, \dots, n+1$. So $\{v_0, \dots, v_n\}$ and $\{w_0, \dots, w_n\}$ are two bases of $K^{n+1}$, hence there exist scalars $\lambda_0, \dots, \lambda_n, \mu_0, \dots, \mu_n$ such that

$$v_{n+1} = \lambda_0 v_0 + \dots + \lambda_n v_n, \quad w_{n+1} = \mu_0 w_0 + \dots + \mu_n w_n,$$

where the coefficients are all different from 0, because of the general position assumption. We replace $v_i$ with $\lambda_i v_i$ and $w_i$ with $\mu_i w_i$ and get two new bases, so there exists a unique automorphism of $K^{n+1}$ transforming the first basis in the second one and, by consequence, also $v_{n+1}$ in $w_{n+1}$. This automorphism induces the required projective transformation on $\mathbb{P}^n$. □

An immediate consequence of the above theorem is that projective subspaces of the same dimension are projectively equivalent. Also two subsets of $\mathbb{P}^n$ formed both by $k$ points in general position are projectively equivalent if $k \leq n + 2$. If $k > n + 2$, this is no longer true, already in the case of four points on a projective line. The problem of describing the classes of projective equivalence of $k$–tuples of points of $\mathbb{P}^n$, for $k > n+2$, is one the first problems of the classical invariant theory. The solution in the case $k = 4$, $n = 1$ is given by the notion of *cross–ratio.*

4. Let $X \subset \mathbb{A}^n$ be an affine variety, then $X_F = X \setminus V(F)$ is isomorphic to a closed subset of $\mathbb{A}^{n+1}$, i.e. to $Y = V(x_{n+1}F - 1, G_1, \dots, G_r)$, where $I(X) = \langle G_1, \dots, G_r \rangle$. Indeed, the following regular maps are inverse each other:

$\phi : X_F \to Y$ such that $(x_1, \dots, x_n) \to (x_1, \dots, x_n, 1/F(x_1, \dots, x_n))$,

$\psi : Y \to X_F$ such that $(x_1, \dots, x_n, x_{n+1}) \to (x_1, \dots, x_n)$.

Hence, $X_F$ is a quasi–projective variety contained in $\mathbb{A}^n$, not closed in $\mathbb{A}^n$, but isomorphic to a closed subset of another affine space.

From now on, the term *affine variety* will denote a quasi–projective variety isomorphic to some affine closed set.

If $X$ is an affine variety and precisely $X \simeq Y$, with $Y \subset \mathbb{A}^n$ closed, then $\mathcal{O}(X) \simeq \mathcal{O}(Y) = K[t_1, \dots, t_n]$ is a finitely generated $K$–algebra. In particular, if $K$ is algebraically closed and $\alpha$ is an ideal strictly contained in $\mathcal{O}(X)$, then $V(\alpha) \subset X$ is non–empty, by the relative form of the Nullstellensatz. From this observation, we can deduce that the quasi–projective variety of next example is not affine.

5. $\mathbb{A}^2 \setminus \{(0,0)\}$ is not affine.

Set $X = \mathbb{A}^2 \setminus \{(0,0)\}$: first of all we will prove that $\mathcal{O}(X) \simeq K[x,y] = \mathcal{O}(\mathbb{A}^2)$, i.e. any regular function on $X$ can be extended to a regular function on the whole plane.

Indeed: let $f \in \mathcal{O}(X)$: if $P \neq Q$ are points of $X$, then there exist polynomials $F, G, F', G'$ such that $f = F/G$ on a neighbourhood $U_P$ of $P$ and $f = F'/G'$ on a neighbourhood $U_Q$ of $Q$. So $F'G = FG'$ on $U_P \cap U_Q \neq \emptyset$, which is open also in $\mathbb{A}^2$, hence dense. Therefore $F'G = FG'$ in $K[x,y]$. We can clearly assume that $F$ and $G$ are coprime and similarly for $F'$ and $G'$. So by the unique factorization property, it follows that $F' = F$ and $G' = G$. In particular $f$ admits a unique representation as $F/G$ on $X$ and $G(P) \neq 0$ for all $P \in X$. Hence $G$ has no zeroes on $\mathbb{A}^2$, so $G = c \in K^*$ and $f \in \mathcal{O}(X)$.

Now, the ideal $\langle x, y \rangle$ has no zeroes in $X$ and is proper: this proves that $X$ is not affine.

We have exploited the fact that a polynomial in more than one variables has infinitely many zeroes, a fact that allows to generalise the previous observation.

On the other hand, the following property holds:

**9.9. Proposition.** *Let $X \subset \mathbb{P}^n$ be quasi–projective. Then $X$ admits an open covering by affine varieties.*

*Proof.* Let $X = X_0 \cup \ldots \cup X_n$ be the open covering of $X$ where $X_i = U_i \cap X = \{P \in X | P[a_0, \dots, a_n], a_i \neq 0\}$. So, fixed $P$, there exists an index $i$ such that

$P \in X_i$. We can assume that $P \in X_0$: $X_0$ is open in some affine variety $Y$ of $\mathbb{A}^n$ (identified with $U_0$); set $X_0 = Y \setminus Y'$, where $Y$, $Y'$ are both closed. Since $P \notin Y'$, there exists $F$ such that $F(P) \neq 0$ and $V(F) \supset Y'$. So $P \in Y \setminus V(F) \subset Y \setminus Y'$ and $Y \setminus V(F)$ is an affine open neighbourhood of $P$ in $Y \setminus Y' = X_0 \subset X$.

□

6. *The Veronese maps.*

Let $n, d$ be positive integers; put $N(n,d) = \binom{n+d}{d} - 1$. Note that $\binom{n+d}{d}$ is equal to the number of (monic) monomials of degree $d$ in the variables $x_0, \dots, x_n$, that is equal to the number of $n+1$–tuples $(i_0, \dots, i_n)$ such that $i_0 + \ldots + i_n = d$, $i_j \geq 0$. Then in $\mathbb{P}^{N(n,d)}$ we can use coordinates $\{v_{i_0 \ldots i_n}\}$, where $i_0, \dots, i_n \geq 0$ and $i_0 + \ldots + i_n = d$. For example: if $n = 2$, $d = 2$, then $N(2,2) = \binom{4}{2} - 1 = 5$. In $\mathbb{P}^5$ we can use coordinates $v_{200}, v_{110}, v_{101}, v_{020}, v_{011}, v_{002}$.

For all $n, d$ we define the map $v_{n,d} : \mathbb{P}^n \to \mathbb{P}^{N(n,d)}$ such that $[x_0, \dots, x_n] \to [v_{d00\ldots0}, v_{d-1,10\ldots0}, \dots, v_{0\ldots00d}]$ where $v_{i_0\ldots i_n} = x_0^{i_0} x_1^{i_1} \ldots x_n^{i_n}$: $v_{n,d}$ is clearly a morphism, its image is denoted $V_{n,d}$ and called *the Veronese variety* of type $(n,d)$. It is in fact the projective variety of equations:

$$(*)\{v_{i_0\ldots i_n} v_{j_0\ldots j_n} - v_{h_0\ldots h_n} v_{k_0\ldots k_n}, \forall i_0 + j_0 = h_0 + k_0, i_1 + j_1 = h_1 + k_1, \ldots$$

We prove this statement in the particular case $n = d = 2$; the general case is similar.

First of all, it is clear that the points of $v_{n,d}(\mathbb{P}^n)$ satisfy the system $(*)$. Conversely, assume that $P[v_{200}, v_{110}, \ldots] \in \mathbb{P}^5$ satisfies the equations $(*)$, which become:

$$\begin{cases} v_{200}v_{020} = v_{110}^2 \\ v_{200}v_{002} = v_{101}^2 \\ v_{002}v_{020} = v_{011}^2 \\ v_{200}v_{011} = v_{110}v_{101} \\ v_{020}v_{101} = v_{110}v_{011} \\ v_{110}v_{002} = v_{011}v_{101} \end{cases}$$

Then, at least one of the coordinates $v_{200}, v_{020}, v_{002}$ is different from 0.

Therefore, if $v_{200} \neq 0$, then $P = v_{2,2}([v_{200}, v_{110}, v_{101}])$; if $v_{020} \neq 0$, then $P = v_{2,2}([v_{110}, v_{020}, v_{011}])$; if $v_{002} \neq 0$, then $P = v_{2,2}([v_{101}, v_{011}, v_{002}])$. Note that, if two of these three coordinates are different from 0, then the points of $\mathbb{P}^2$ found in this way have proportional coordinates, so they coincide.

We have also proved in this way that $v_{2,2}$ is an isomorphism between $\mathbb{P}^2$ and $V_{2,2}$, called the Veronese surface of $\mathbb{P}^5$. The same happens in the general case.

If $n = 1$, $v_{1,d} : \mathbb{P}^1 \to \mathbb{P}^d$ takes $[x_0, x_1]$ to $[x_0^d, x_0^{d-1}x_1, \dots, x_1^d]$: the image is called the *rational normal curve* of degree $d$, it is isomorphic to $\mathbb{P}^1$. If $d = 3$, we find the skew cubic.

Let now $X \subset \mathbb{P}^n$ be a hypersurface of degree $d$: $X = V_P(F)$, with

$$F = \sum_{i_0 + \ldots + i_n = d} a_{i_0\ldots i_n} x_0^{i_0} \ldots x_n^{i_n}.$$

Then $v_{n,d}(X) \simeq X$: it is the set of points

$$\{v_{i_0\dots i_n} \in \mathbb{P}^{N(n,d)} | \sum_{i_0+\dots+i_n=d} a_{i_0\dots i_n} v_{i_0\dots i_n} = 0 \text{ and } [v_{i_0\dots i_n}] \in V_{n,d}\}.$$

It coincides with $V_{n,d} \cap H$, where $H$ is a hyperplane of $\mathbb{P}^{N(n,d)}$: a hyperplane section of the Veronese variety. This is called the linearisation process, allowing to "transform" a hypersurface in a hyperplane, modulo the Veronese isomorphism.

The Veronese surface $V$ of $\mathbb{P}^5$ enjoys a lot of interesting properties. Most of them follow from its property of being covered by a 2-dimensional family of conics, which are precisely the images via $v_{2,2}$ of the lines of the plane.

To see this, we'll use as coordinates in $\mathbb{P}^5$ $w_{00}, w_{01}, w_{02}, w_{11}, w_{12}, w_{22}$, so that $v_{2,2}$ sends $[x_0, x_1, x_2]$ to the point of coordinates $w_{ij} = x_i x_j$. With this choice of coordinates, the equations of $V$ are obtained by annihilating the $2 \times 2$ minors of the symmetric matrix:

$$M = \begin{pmatrix} w_{00} & w_{01} & w_{02} \\ w_{01} & w_{11} & w_{12} \\ w_{02} & w_{12} & w_{22} \end{pmatrix}$$

Let $\ell$ be a line of $\mathbb{P}^2$ of equation $b_0x_0 + b_1x_1 + b_2x_2 = 0$. Its image is the set of points of $\mathbb{P}^5$ with coordinates $w_{ij} = x_i x_j$, such that there exists a non-zero triple $[x_0, x_1, x_2]$ with $b_0x_0 + b_1x_1 + b_2x_2 = 0$. But this last equation is equivalent to the system:

$$\begin{cases} b_0x_0^2 + b_1x_0x_1 + b_2x_0x_2 = 0 \\ b_0x_0x_1 + b_1x_1^2 + b_2x_1x_2 = 0 \\ b_0x_0x_2 + b_1x_1x_2 + b_2x_2^2 = 0 \end{cases}$$

It represents the intersection of $V$ with the plane

$$(*) \begin{cases} b_0w_{00} + b_1w_{01} + b_2w_{02} = 0 \\ b_0w_{01} + b_1w_{11} + b_2w_{12} = 0 \\ b_0w_{02} + b_1w_{12} + b_2w_{22} = 0 \end{cases},$$

so $v_{2,2}(\ell)$ is a plane curve. Its degree is the number of points in its intersection with a general hyperplane in $\mathbb{P}^5$: this corresponds to the intersection in $\mathbb{P}^2$ of $\ell$ with a conic (a hypersurface of degree 2). Therefore $v_{2,2}(\ell)$ is a conic.

So the isomorphism $v_{2,2}$ transforms the geometry of the lines in the plane in the geometry of the conics on the Veronese surface. In particular, given two distinct points on $V$, there is exactly one conic contained in $V$ and passing through them.

From this observation it is easy to deduce that the *secant lines* of $V$, i.e. the lines meeting $V$ at two points, are precisely the lines of the planes generated by the conics contained in $V$, so that the (closure of the) union of these secant lines coincides with the union of the planes of the conics of $V$. This union results to be the cubic hypersurface defined by the equation

$$\det M = \det \begin{pmatrix} w_{00} & w_{01} & w_{02} \\ w_{01} & w_{11} & w_{12} \\ w_{02} & w_{12} & w_{22} \end{pmatrix} = 0.$$

Indeed a point of $\mathbb{P}^5$, of coordinates $[w_{ij}]$ belongs to the plane of a conic contained in $V$ if and only if there exists a non-zero triple $[b_0, b_1, b_2]$ which is solution of the homogeneous system (*).

### b) Rational maps

Let $X, Y$ be quasi–projective varieties.

**9.10. Definition.** The *rational maps* from $X$ to $Y$ are the germs of regular maps from open subsets of $X$ to $Y$, i.e. equivalence classes of pairs $(U, \phi)$, where $U \neq \emptyset$ is open in $X$ and $\phi : U \to Y$ is regular, with respect to the relation: $(U, \phi) \sim (V, \psi)$ if and only if $\phi|_{U\cap V} = \psi|_{U\cap V}$. The following Lemma guarantees that the above defined relation satisfies the transitive property.

**9.11. Lemma.** *Let $\phi, \psi : X \to Y \subset \mathbb{P}^n$ be regular maps between quasi-projective varieties. If $\phi|_U = \psi|_U$ for $U \subset X$ open and non–empty, then $\phi = \psi$.*

*Proof.* Let $P \in X$ and consider $\phi(P)$, $\psi(P) \in Y$. There exists a hyperplane $H$ such that $\phi(P) \notin H$ and $\psi(P) \notin H$ (otherwise the dual projective space $\check{\mathbb{P}}^n$ would be the union of its two hyperplanes consisting of hyperplanes of $\mathbb{P}^n$ passing through $\phi(P)$ and $\psi(P)$). Up to a projective transformation, we can assume that $H = V_P(x_0)$, so $\phi(P), \psi(P) \in U_0$. Set $V = \phi^{-1}(U_0) \cap \psi^{-1}(U_0)$: an open neighbourhood of $P$. Consider the restrictions of $\phi$ and $\psi$ from $V$ to $Y \cap U_0$: they are regular maps which coincide on $V \cap U$, hence their coordinates $\phi_i$, $\psi_i$, $i = 1, \dots, n$, coincide on $V \cap U$, hence on $V$. So $\phi_i|_V = \psi_i|_V$. In particular $\phi(P) = \psi(P)$. □

A rational map from $X$ to $Y$ will be denoted $\phi : X \dashrightarrow Y$. As for rational functions, the domain of definition of $\phi$, dom $\phi$, is the maximum open subset of $X$ such that $\phi$ is regular at the points of dom $\phi$.

The following proposition follows from the characterization of rational functions on affine varieties.

**9.12. Proposition.** *Let $X, Y$ be affine algebraic sets, with $Y$ closed in $\mathbb{A}^n$. Then $\phi : X \dashrightarrow Y$ is a rational map if and only if $\phi = (\phi_1, \dots, \phi_n)$, where $\phi_1, \dots, \phi_n \in K(X)$.* □

If $X \subset \mathbb{P}^n$, $Y \subset \mathbb{P}^m$, then a rational map $X \dashrightarrow Y$ is assigned by giving $m+1$

homogeneous polynomials of $K[x_0, x_1, \dots, x_n]$ of the same degree, $F_0, \dots, F_m$, such that *at least one* of them is not identically zero on $X$.

A rational map $\phi : X \dashrightarrow Y$ is called *dominant* if the image of $X$ via $\phi$ is dense in $X$, i.e. if $\overline{\phi(U)} = X$, where $U = \text{dom } \phi$. If $\phi : X \dashrightarrow Y$ is dominant and $\psi : Y \dashrightarrow Z$ is any rational map, then dom $\psi \cap \text{Im}\phi \neq \emptyset$, so we can define $\psi \circ \phi : X \dashrightarrow Z$: it is the germ of the map $\psi \circ \phi$, regular on $\phi^{-1}(\text{dom } \psi \cap \text{Im}\phi)$.

**9.13. Definition.** A *birational map* from $X$ to $Y$ is a rational map $\phi : X \dashrightarrow Y$ such that $\phi$ is dominant and there exists $\psi : Y \dashrightarrow X$, a dominant rational map, such that $\psi \circ \phi = 1_X$ and $\phi \circ \psi = 1_Y$ as rational maps. In this case, $X$ and $Y$ are called *birationally equivalent* or simply *birational*.

If $\phi : X \dashrightarrow Y$ is a dominant rational map, then we can define the comorphism $\phi^* : K(Y) \to K(X)$ in the usual way: it is an injective $K$–homomorphism.

**9.14. Proposition.** *Let $X$, $Y$ be quasi–projective varieties, $u : K(Y) \to K(X)$ be a $K$–homomorphism. Then there exists a rational map $\phi : X \dashrightarrow Y$ such that $\phi^* = u$.*

*Proof.* $Y$ is covered by open affine varieties $Y_\alpha$, $\alpha \in I$ (by Proposition 9.9): for all index $\alpha$, $K(Y) \simeq K(Y_\alpha)$ (Prop. 8.8) and $K(Y_\alpha) \simeq K(t_1, \dots, t_n)$, where $t_1, \dots, t_n$ can be interpreted as coordinate functions on $Y_\alpha$. Then $u(t_1), \dots, u(t_n) \in K(X)$ and there exists $U \subset X$, non–empty open subset such that $u(t_1), \dots, u(t_n)$ are all regular on $U$. So $u(K[t_1, \dots, t_n]) \subset \mathcal{O}(U)$ and we can consider the regular map $u^\sharp : U \to Y_\alpha \hookrightarrow Y$. The germ of $u^\sharp$ gives a rational map $X \dashrightarrow Y$. It is possible to check that this rational map does not depend on the choice of $Y_\alpha$ and $U$. □

**9.15. Theorem.** *Let $X$, $Y$ be quasi–projective varieties. The following are equivalent:*

*(i) $X$ is birational to $Y$;*

*(ii) $K(X) \simeq K(Y)$;*

*(iii) there exist non–empty open subsets $U \subset X$ and $V \subset Y$ such that $U \simeq V$.*

*Proof.*

(i) $\Leftrightarrow$ (ii) via the construction of the comorphism $\phi^*$ associated to $\phi$ and of $u^\sharp$, associated to $u : K(Y) \to K(X)$. One checks that both constructions are functorial.

(i) $\Rightarrow$ (iii) Let $\phi : X \dashrightarrow Y$, $\psi : Y \dashrightarrow X$ be inverse each other. Put $U' = \text{dom } \phi$ and $V' = \text{dom } \psi$. By assumption, $\psi \circ \phi$ is defined on $\phi^{-1}(V')$ and coincides with $1_X$ there. Similarly, $\psi \circ \phi$ is defined on $\psi^{-1}(U')$ and equal to $1_Y$. Then $\phi$ and $\psi$ establish an isomorphism between the corresponding sets $U := \phi^{-1}(\psi^{-1}(U'))$ and $V := \psi^{-1}(\phi^{-1}(V'))$.

(iii) $\Rightarrow$ (ii) $U \simeq V$ implies $K(U) \simeq K(V)$; but $K(U) \simeq K(X)$ and $K(V) \simeq K(Y)$ (Prop.8.8), so $K(X) \simeq K(Y)$ by transitivity. □

**9.16. Corollary.** *If $X$ is birational to $Y$, then* $\dim X = \dim Y$. □

**9.17. Examples.**

a) The cuspidal cubic $Y = V(x^3 - y^2) \subset \mathbb{A}^2$.

We have seen that $Y$ is not isomorphic to $\mathbb{A}^1$, but in fact $Y$ and $\mathbb{A}^1$ are birational. Indeed, the regular map $\phi : \mathbb{A}^1 \to Y$, $t \to (t^2, t^3)$, admits a rational inverse $\psi : Y \dashrightarrow \mathbb{A}^1$, $(x, y) \to \frac{y}{x}$. $\psi$ is regular on $Y \setminus \{(0,0)\}$, $\psi$ is dominant and $\psi \circ \phi = 1_{\mathbb{A}^1}$, $\phi \circ \psi = 1_Y$ as rational maps. In particular, $\phi^* : K(Y) \to K(X)$ is a field isomorphism. Recall that $K[Y] = K[t_1, t_2]$, with $t_1^2 = t_2^3$, so $K(Y) = K(t_1, t_2) = K(t_2/t_1)$, because $t_1 = (t_2/t_1)^2 = t_2^2/t_1^2 = t_1^3/t_1^2$ and $t_2 = (t_2/t_1)^3 = t_2^3/t_1^3 = t_2^3/t_2^2$, so $K(Y)$ is generated by a unique transcendental element. Notice that $\phi$ and $\psi$ establish isomorphisms between $\mathbb{A}^1 \setminus \{0\}$ and $Y \setminus \{(0,0)\}$.

b)*Rational maps from $\mathbb{P}^1$ to $\mathbb{P}^n$.*

Let $\phi : \mathbb{P}^1 \dashrightarrow \mathbb{P}^n$ be rational: on some open $U \subset \mathbb{P}^1$,

$$\phi([x_0, x_1]) = [F_0(x_0, x_1), \dots, F_n(x_0, x_1)],$$

with $F_0, \dots, F_n$ homogeneous of the same degree, without non–trivial common factors. Assume that $F_i(P) = 0$ for a certain index $i$, with $P = [a_0, a_1]$. Then $F_i \in I_h(P) = \langle a_1 x_0 - a_0 x_1 \rangle$, i.e. $a_1 x_0 - a_0 x_1$ is a factor of $F_i$. This remark implies that $\forall\ Q \in \mathbb{P}^1$ there exists $i \in \{0, \dots, n\}$ such that $F_i(Q) \neq 0$, because otherwise $F_0, \dots, F_n$ would have a common factor of degree 1. Hence we conclude that $\phi$ is regular.

We have obtained that any rational map from $\mathbb{P}^1$ is in fact regular.

c) *Projections.*

Let $\phi : \mathbb{P}^n \dashrightarrow \mathbb{P}^m$ be given in matrix form by $Y = AX$, where $A$ is a $(m+1) \times (n+1)$-matrix, with entries in $K$. Then $\phi$ is a rational map, regular on $\mathbb{P}^n \setminus \mathbb{P}(\mathrm{Ker} A)$. Put $\Lambda := \mathbb{P}(\mathrm{Ker} A)$. If $A = (a_{ij})$, this means that $\Lambda$ has cartesian equations

$$\begin{cases} a_{00}x_0 + \ldots + a_{0n}x_n = 0 \\ a_{10}x_0 + \ldots + a_{1n}x_n = 0 \\ \ldots \\ a_{m0}x_0 + \ldots + a_{mn}x_n = 0 \end{cases}$$

The map $\phi$ has a geometric interpretation: it can be seen as the *projection of centre* $\Lambda$ to a complementar linear space. First of all, we can assume that rk $A = m+1$, otherwise we replace $\mathbb{P}^m$ with $\mathbb{P}(\mathrm{Im}\ A)$; hence $\dim \Lambda = n - (m+1)$.

Consider first the case $\Lambda : x_0 = \ldots = x_m = 0$; we identify $\mathbb{P}^m$ with the subspace of $\mathbb{P}^n$ of equations $x_{m+1} = \ldots = x_n = 0$, so $\Lambda$ and $\mathbb{P}^m$ are complementar subspaces, i.e. $\Lambda \cap \mathbb{P}^m = \emptyset$ and the linear span of $\Lambda$ and $\mathbb{P}^m$ is $\mathbb{P}^n$. Then, for $Q \in \mathbb{P}^n \setminus \Lambda$, $\phi(Q) = [x_0, \dots, x_m, 0, \dots, 0]$: it is the intersection of $\mathbb{P}^m$ with the linear span of $\Lambda$ and $Q$. In fact, if $Q[a_0, \dots, a_n]$ then $\overline{\Lambda Q}$ has equations

$$\{a_i x_j - a_j x_i = 0, i, j = 0, \dots, m \quad \text{(check!)}$$

so $\overline{\Lambda Q}\cap\mathbb{P}^m$ has coordinates $[a_0,\dots,a_m,0,\dots,0]$.

In the general case, if $\Lambda = V_P(L_0,\dots,L_m)$, with $L_0,\dots,L_m$ linearly independent forms, we can identify $\mathbb{P}^m$ with $V_P(L_{m+1},\dots,L_n)$, where $L_0,\dots,L_m$, $L_{m+1},\dots,L_n$ is a basis of $(K^{n+1})^*$. Then $L_0,\dots,L_m$ can be interpreted as coordinate functions on $\mathbb{P}^m$.

If $m = n-1$, then $\Lambda$ is a point $P$ and $\phi$, often denoted $\pi_P$, is the projection from $P$ to a hyperplane not containing $P$.

d)*Rational and unirational varieties.*

A quasi–projective variety $X$ is called *rational* if it is birational to a projective space $\mathbb{P}^n$, or equivalently to $\mathbb{A}^n$. Indeed, in view of Thereom 9.15 (*iii*), $\mathbb{P}^n$ and $\mathbb{A}^n$ are birationally equivalent.

By Theorem 9.15, $X$ is rational if and only if $K(X)\simeq K(\mathbb{P}^n) = K(x_1,\dots,x_n)$ for some $n$, i.e. $K(X)$ is an extension of $K$ generated by a transcendence basis (a purely transcendental extension of $K$). In an equivalent way, $X$ is rational if there exists a rational map $\phi:\mathbb{P}^n \dashrightarrow X$ which is dominant and is an isomorphism if restricted to a suitable open subset $U\subset\mathbb{P}^n$. Hence $X$ admits a *birational parametrization* by polynomials in $n$ parameters.

A weaker notion is that of *unirational* variety: $X$ is unirational if there exists a dominant rational map $\mathbb{P}^n \dashrightarrow X$ i.e. if $K(X)$ is contained in the quotient field of a polynomial ring. Hence $X$ can be parametrised by polynomials, but not necessarily generically one–to–one.

It is clear that, if $X$ is rational, then it is unirational. The converse implication has been an important open problem, up to 1971, when it has been solved in the negative, for varieties of dimension $\geq 3$ (Clemens–Griffiths and Iskovskih–Manin). Nevertheless rationality and unirationality are equivalent for curves (Theorem of Lüroth, 1880) and for surfaces if char$K = 0$ (Theorem of Castelnuovo, 1894).

As an example of rational variety with an explicit rational parametrization constructed geometrically, let us consider the following quadric of maximal rank in $\mathbb{P}^3$: $X = V_P(x_0x_3 - x_1x_2)$, an irreducible hypersurface of degree 2. Let $\pi_P:\mathbb{P}^3 \dashrightarrow \mathbb{P}^2$ be the projection of centre $P[1,0,0,0]$, such that $\pi_P([y_0,y_1,y_2,y_3]) = [y_1,y_2,y_3]$. The restriction of $\pi_P$ to $X$ is a rational map $\tilde{\pi}_P : X \dashrightarrow \mathbb{P}^2$, regular on $X\setminus\{P\}$. $\tilde{\pi}_P$ has a rational inverse: indeed consider the rational map $\psi:\mathbb{P}^2 \dashrightarrow X$, $[y_1,y_2,y_3]\to[y_1y_2,y_1y_3,y_2y_3,y_3^2]$. The equation of $X$ is satisfied by the points of $\psi(\mathbb{P}^2)$: $(y_1y_2)y_3^2 = (y_1y_3)(y_2y_3)$. $\psi$ is regular on $\mathbb{P}^2\setminus V_P(y_1y_2,y_3)$. Let us compose $\psi$ and $\tilde{\pi}_P$:

$$[y_0,\dots,y_3]\in X \overset{\pi_P}{\to} [y_1,y_2,y_3] \overset{\psi}{\to} [y_1y_2,y_1y_3,y_2y_3,y_3^2];$$

$y_1y_2 = y_0y_3$ implies $\psi\circ\pi_P = 1_X$. In the opposite order:

$$[y_1,y_2,y_3] \overset{\psi}{\to} [y_1y_2,y_1y_3,y_2y_3,y_3^2] \overset{\pi_P}{\to} [y_1y_3,y_2y_3,y_3^2] = [y_1,y_2,y_3].$$

So $X$ is birational to $\mathbb{P}^2$ hence it is a rational surface.

Note that if we consider a projection $\pi_P$ whose centre $P$ is not on the quadric, we get a regular $2:1$ map to the plane, certainly not birational.

e) *A birational non–regular map from* $\mathbb{P}^2$ *to* $\mathbb{P}^2$.

The following rational map is called the *standard quadratic map*:

$$Q : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2, \quad [x_0, x_1, x_2] \to [x_1x_2, x_0x_2, x_0x_1].$$

$Q$ is regular on $U := \mathbb{P}^2 \setminus \{A, B, C\}$, where $A[1,0,0]$, $B[0,1,0]$, $C[0,0,1]$ are the fundamental points (see Fig. 2)

Let $a$ be the line through $B$ and $C$: $a = V_P(x_0)$, and similarly $b = V_P(x_1)$, $c = V_P(x_2)$. Then $Q(a) = A$, $Q(b) = B$, $Q(c) = C$. Outside these three lines $Q$ is an isomorphism. Precisely, put $U' = \mathbb{P}^2 \setminus \{a \cup b \cup c\}$; then $Q : U' \to \mathbb{P}^2$ is regular, the image is $U'$ and $Q^{-1} : U' \to U'$ coincides with $Q$. Indeed,

$$[x_0, x_1, x_2] \stackrel{Q}{\to} [x_1x_2, x_0x_2, x_0x_1] \stackrel{Q}{\to} [x_0^2x_1x_2, x_0, x_1^2x_2, x_0x_1x_2^2].$$

So $Q \circ Q = 1_{\mathbb{P}^2}$ as rational map, hence $Q$ is birational and $Q = Q^{-1}$.

– Fig. 2 –

The set of the birational maps $\mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ is a group, called the *Cremona group*. At the end of XIX century, Max Noether proved that the Cremona group is generated by $PGL(3, K)$ and by the single standard quadratic map above. The analogous groups for $\mathbb{P}^n$, $n \geq 3$, are much more complicated and a complete description is still unknown.

We conclude this section with a theorem illustrating an application of the linearisation procedure. We shall use the following notation: given a homogeneous polynomial $F \in K[x_0, x_1, \ldots, x_n]$, $D(F) := \mathbb{P}^n \setminus V_P(F)$.

**9.16. Theorem.** *Let $W \subset \mathbb{P}^n$ be a closed projective variety. Let $F$ be a homogeneous polynomial of degree $d$ in $K[x_0, x_1, \dots, x_n]$ such that $W \not\subseteq V_P(F)$. Then $W \cap D(F)$ is an affine variety.*

*Proof.* The assumption $W \not\subseteq V_P(F)$ is equivalent to $W \cap D(F) \neq \emptyset$. Let us consider the $d$-tuple Veronese embedding $v_{n,d} : \mathbb{P}^n \to \mathbb{P}^{N(n,d)}$, with $N(n,d) = \binom{n+d}{d} - 1$, that gives the isomorphism $\mathbb{P}^n \simeq V_{n,d}$. In this isomorphism the hypersurface $V_P(F)$ corresponds to a hyperplane section $V_{n,d} \cap H$, for a suitable hyperplane $H$ in $\mathbb{P}^{N(n,d)}$. Therefore we have $W \cap D(F) \simeq v_{n,d}(W \cap D(F)) = v_{n,d}(W) \setminus H = v_{n,d}(W) \cap (\mathbb{P}^{N(n,d)} \setminus H)$. There exists a projective isomorphism $\tau : \mathbb{P}^{N(n,d)} \to \mathbb{P}^{N(n,d)}$ such that $\tau(H) = H_0$, the fundamental hyperplane of equation $x_0 = 0$. Therefore, denoting $X := v_{n,d}(W)$, we get $X \cap (\mathbb{P}^{N(n,d)} \setminus H) \simeq \tau(X) \cap (\mathbb{P}^{N(n,d)} \setminus H_0) = \tau(X) \cap U_0$, which proves the theorem. □

As a consequence of Theorem 9.16, we get that the open subsets of the form $W \cap D(F)$ form a topology basis of affine varieties for $W$.

**Exercises to §9.**

1. Let $\phi : \mathbb{A}^1 \to \mathbb{A}^n$ be the map defined by $t \to (t, t^2, \dots, t^n)$.

a) Prove that $\phi$ is regular and describe $\phi(\mathbb{A}^1)$;

b) prove that $\phi : \mathbb{A}^1 \to \phi(\mathbb{A}^1)$ is an isomorphism;

c) give a description of $\phi^*$ and $\phi^{-1*}$.

2. Let $f : \mathbb{A}^2 \to \mathbb{A}^2$ be defined by: $(x, y) \to (x, xy)$.

a) Describe $f(\mathbb{A}^2)$ and prove that it is not locally closed in $\mathbb{A}^2$.

b) Prove that $f(\mathbb{A}^2)$ is a constructible set in the Zariski topology of $\mathbb{A}^2$ (i.e. a finite union of locally closed sets).

3. Prove that the Veronese variety $V_{n,d}$ is not contained in any hyperplane of $\mathbb{P}^{N(n,d)}$.

4. Let $GL_n(K)$ be the set of invertible $n \times n$ matrices with entries in $K$. Prove that $GL_n(K)$ can be given the structure of an affine variety.

5. Show the unicity of the projective transformation $\tau$ of Theorem 9.8.

6. Let $\phi : X \to Y$ be a regular map and $\phi^*$ its comorphism. Prove that the kernel of $\phi^*$ is the ideal of $\phi(X)$ in $\mathcal{O}(Y)$. In the affine case, deduce that $\phi$ is dominant if and only if $\phi^*$ is injective.

7. Prove that $\mathcal{O}(X_F)$ is isomorphic to $\mathcal{O}(X)_f$, where $X$ is an affine algebraic variety, $F$ a polynomial and $f$ the function on $X$ defined by $F$.