Appendix: An axiomatization for classical ZF

We propose here a first-order axiomatization of the Zermelo-Fraenkel set theory. Our formulation of the axioms (cf. [10]) slightly differs from, but is equivalent to, versions of this theory which can be found in the literature.

(E) $\forall x\, \forall y\, \exists d\, \big((d \in x \Leftrightarrow d \in y) \Longrightarrow x = y \big)$

(D) $\forall x\, \forall y\, \exists d \big(y \in d \,\&\, \forall v\, (v = x \Leftrightarrow \exists w \in d\; v \in w \;\&\; \exists \ell \in d\; v \notin \ell\,) \big)$

(P) $\forall x\, \exists p\, \forall y \big((\forall v \in y\; v \in x) \Longrightarrow y \in p \big)$

(T) $\forall x\, \exists t\, (x \in t \,\&\, \forall v \in t\, \forall y \in v\; y \in t\,)$

(S) $\forall a\, \exists b\, \forall c\, \Big(c \in b \Leftrightarrow \exists d\, \big(\forall x\, (\varphi[a,x] \Leftrightarrow x = d\,) \,\&\, c \in d \,\&\, \psi[a,c]\,\big) \Big)$

(S′) $\forall a'\, \forall a\, \exists b\, \forall c\, \Big(\exists e \in a'\, \forall x\, \big(\chi[e,a,x] \Leftrightarrow x = c \big) \Longrightarrow c \in b \Big)$

(I) $\forall x\, \exists i\, \big(x \in i \,\&\, \forall y \in i\, \exists u \in i\, \forall z\, (z \in u \Leftrightarrow z = y\,) \big)$

(R) $\forall x\, \exists m\, \forall y\; \big(y \in x \Longrightarrow m \in x \,\&\, y \notin m\,\big)$

(C) $\forall x\, \big(\forall p \in x\, \exists!\, q \in x\, \exists z \in p\;\; z \in q \;\Longrightarrow\; \exists c\, \forall r \in x\, \exists!\, k \in c\;\; k \in r\,\big)$

Roughly cast in words, this is the content of each postulate:

(E) *Extensionality:* If two sets differ, one has a member not owned by the other.

(D) *Elementary sets:* An empty set exists; one can adjoin any set x as a new member to any set y, thereby getting a set w; one can remove from a set y any one of its members, thereby getting a set ℓ. (Cf. [11].)

(P) *Powerset:* For any set x, there is a set to which all subsets of x belong.

(T) *Transitive closure:* Any set x belongs to a *full* set, namely to a set t whose elements are also subsets of t.

(S) *Subsets:* To every set a, there corresponds a set b which is null unless there is exactly one d fulfilling $\varphi[a, d]$, and which in the latter case consists of all elements c of d for which $\psi[a, c]$ holds.

(S′) *Replacement:* To every pair a, a' of sets there corresponds a set comprising the images, under the functional part of $\chi[e, a, d]$, of all pairs e, a with e belonging to a'.

(I) *Infinity:* For any set x, one can form a set i to which x belongs, owning as a member, along with every y that belongs to it, the singleton set $\{y\}$. (Trivially i is infinite when x is not a singleton).[12]

(R) *Regularity:* Membership is well-founded.

(C) *Choice:* Every set x constituted by non-empty pairwise disjoint sets admits a 'choice' set, i.e., a set c whose intersection with any element of x is singleton.

As we have discussed in Sec. 3, it suffices to replace the pair **(R)**, **(E)** of axioms by **(AFA)** in order to get a hyperset theory closely analogous (but antithetic) to ZF; on the other hand, when **(R)** is available one can simplify **(I)** into

$$\textbf{(I}'\textbf{)} \quad \exists x \, \exists i \, (\, x \in i \; \& \; \forall y \in i \, \exists u \in i \;\; y \in u \,).$$

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[12] The following alternative version of the infinity axiom, which deserves some interest, was proposed in [24]:

$$\exists a \, \exists b \Big(a \neq b \; \& \; a \notin b \; \& \; b \notin a \; \& \; \forall x \in a \, \forall y \in b \, (y \in x \vee x \in y) \; \& \\ \forall x \in a \, \forall y \in x \; y \in b \; \& \; \forall x \in b \, \forall y \in x \; y \in a \; \& \; \forall x \in a \; x \notin b \Big).$$

Un assioma di finitezza, antitetico all'assioma dell'infinito presente nella teoria di Zermelo-Fraenkel

Eugenio Omodeo

29 aprile 2019

In un articolo del 1924 da titolo *Sur les ensembles fini*, Alfred Tarski propone l'assioma

$$\textbf{(F)}\quad \forall\, k\, \forall\, f \in k\; \exists\, a \in k\; \forall\, b \in k\; (\overbrace{\forall\, d \in b\;\, d \in a}^{b \subseteq a} \Longrightarrow b = a)\,.$$

Lèggi: "Qualsiasi insieme k abbia almeno un elemento, f, ne possiede anche uno, a, che in k è minimale rispetto alla relazione $\subseteq$ d'inclusione tra insiemi".