Sia $u(x,y) = \log\sqrt{x^2+y^2}$

Se esistesse $v = v(x,y)$ $(x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}$

t.c.

$$v_x = -u_y$$

$$v_y = u_x$$

avremmo: $v_x = -\dfrac{y}{x^2+y^2} = -\dfrac{\sin\vartheta}{r}$

$$v_y = \frac{x}{x^2+y^2} = \frac{\cos\vartheta}{r}$$

$$\Rightarrow \quad \boxed{v(x,y) = \vartheta} = \vartheta + 2k\pi$$

$$\partial_\theta = \frac{\partial x}{\partial\theta}\partial_x + \frac{\partial y}{\partial\theta}\partial_y = \frac{-r\sin\vartheta\,\partial_x + r\cos\vartheta\,\partial_y}{r}$$

$$\partial_\theta v = 1$$

$$\partial_r = \frac{\partial x}{\partial r}\partial_x + \frac{\partial y}{\partial r}\partial_y = \cos\vartheta\,\partial_x + \sin\vartheta\,\partial_y$$

$$\partial_r v = 0 \qquad v = v(\vartheta)$$

$$v(x,y) = \vartheta + 2k\pi \qquad \forall k$$

$f \in H(\Omega)$, $f = u + iv$

$$\begin{cases} v_x = -u_y \\ v_y = u_x \end{cases}$$

$$\Delta u = \Delta v = 0$$

<u>Def.</u> v si dice armonica coniugata di u.

<u>Oss</u> Se v è coniugata a u, allora u è coniugata a $-v$

$$g = if = i(u+iv) = -v + i\underline{u}$$

Teor. Se Ω è sempl. connesso e $u \in C^2(\Omega, \mathbb{R})$ t.c.

$$\Delta u = 0 \quad \text{in } \Omega$$

allora esiste una armonica coniugata v di u in Ω.

Caso particolare: Ω convesso.

Teor. Sia Ω convesso, $f \in H(\Omega)$ allora esiste $F \in H(\Omega)$ t.c. $F' = f$.

$$\Delta = \partial_x^2 + \partial_y^2$$

$$\partial_x = \frac{\partial z}{\partial x} \partial_z + \frac{\partial \bar{z}}{\partial x} \partial_{\bar{z}} = \partial_z + \partial_{\bar{z}}$$

$$\partial_y = \frac{\partial z}{\partial y} \partial_z + \frac{\partial \bar{z}}{\partial y} \partial_{\bar{z}} = i \partial_z - i \partial_{\bar{z}}$$

$$\Delta = 4\partial_z\partial_{\bar z} = 4\partial_{\bar z}\partial_z$$

Sia $u \in C^2(\Omega)$ t.c. $\Delta u = 0$

$$4\,\underline{\underline{\partial_{\bar z}}}(\partial_z u) = 0 \quad \text{in } \Omega.$$

$$\partial_z u \in H(\Omega)$$

Se Ω è convesso $\exists\, \varphi \in H(\Omega)$ t.c.

$$\varphi' = \partial_z u \quad , \quad \partial_z(\varphi - u) = 0$$

$$\partial_{\bar z}\left(\overline{\varphi - u}\right) = 0 \quad , \quad \overline{\varphi - u} \in H(\Omega)$$

Sia $\psi = \overline{(\varphi - u)}$, $\bar\psi = \varphi - u$

$$u = \underline{\varphi - \overline{\psi}} \quad , \ u \text{ è a valori reali}$$

cioè $\quad u = \bar{u}$

$$\varphi - \overline{\psi} = \bar{\varphi} - \psi$$

$$u = \left(\varphi - \overline{\psi} + \bar{\varphi} - \psi\right)\frac{1}{2} = \frac{\varphi - \psi + \overline{(\varphi - \psi)}}{2}$$

$$f = \varphi - \psi \in H(\Omega)$$

$$u = \mathrm{Re}\, f$$

se prendiamo $v = \mathrm{Im}\, f$, v è l'arm. coniugata di u $\square$

Def. Sia γ una curva reg. a tratti

sia $\varphi \in C(\gamma^*, \mathbb{R})$ poniamo

$$\int_\gamma \varphi\, \underline{|dz|} = \underline{\int_a^b \varphi(\gamma(t))\, \overbrace{|\gamma'(t)|\, dt}^{ds}}$$

dove $\gamma : [a,b] \to \mathbb{R}^2$, $\gamma^* = \gamma([a,b])$.

Posto $L(\gamma) = \int_a^b |\gamma'| \, dt$

Se $|\varphi| \leq M$ su γ^*

$$\left| \int_\gamma \varphi \, |dz| \right| \leq M \, L(\gamma)$$

— · —

<u>Teor.</u> (Formula di Poisson, per funz. olomorfe).

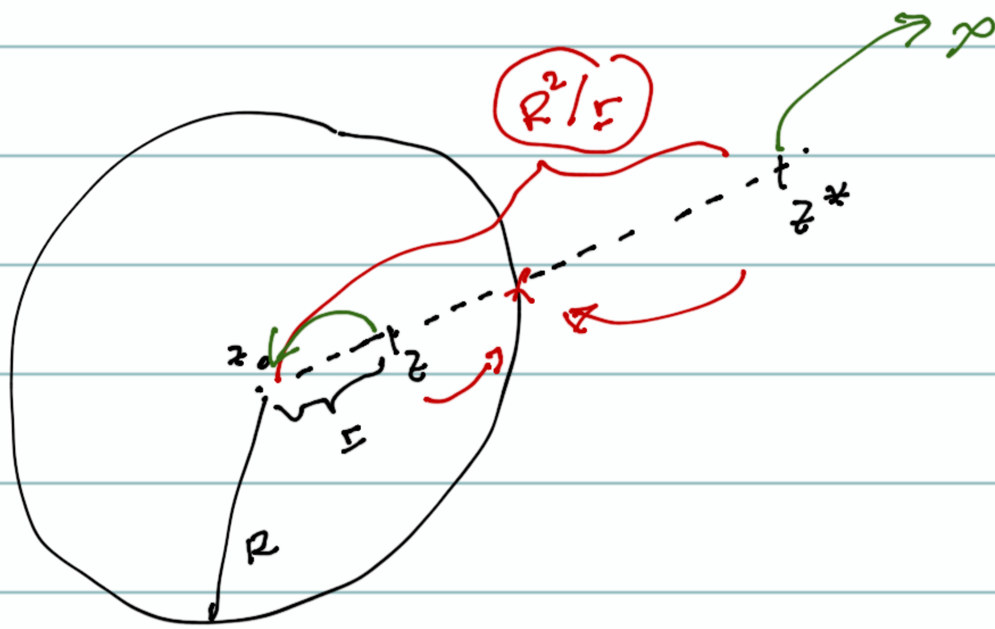
Siano $R, \tilde{R}$, $0 < \underline{R} < \tilde{R}$, e sia $f \in H(B_{\tilde{R}}(z_0))$

$\forall z \in B_R(z_0)$ vale

$$f(z) = \frac{1}{\underbrace{2\pi R}_{\text{reale}}} \int_{\partial B_R(z_0)} \overbrace{\boxed{\frac{R^2 - |z - z_0|^2}{|\xi - z|^2}}}^{\text{a valori reali}} f(\xi) \, |d\xi|$$

Dim

$$\frac{1}{2\pi i}\int_{\partial B_R(z_0)} \frac{f(\xi)}{\xi - z}\, d\xi = \begin{cases} f(z) & \text{se } z \in B_R(z_0) \\ 0 & \text{se } z \in \mathbb{C} \setminus \overline{B_R(z_0)} \end{cases}$$



Poniamo

$$z^* = z_0 + \frac{R^2}{\overline{(z - z_0)}}$$

(inversione per raggi vettori)

$$f(z) = \frac{1}{2\pi i}\int_{\partial B_R(z_0)} \frac{f(\xi)}{\xi - z}\, d\xi - \frac{1}{2\pi i}\int_{\partial B_R(z_0)} \frac{f(\xi)}{\xi - z^*}\, d\xi$$

$$= \frac{1}{2\pi i} \int_{\partial B_R(z_0)} \left(\frac{1}{\xi - z} - \frac{1}{\xi - z^*} \right) f(\xi)\, d\xi$$

$$\xi = z_0 + R e^{i\vartheta} \quad , \quad d\xi = i R e^{i\vartheta} d\vartheta =$$

$$= i \frac{(\xi - z_0)}{R} |d\xi|$$

$$f(z) = \frac{1}{2\pi R} \int_{\partial B_R(z_0)} \overbrace{\left(\frac{1}{\xi - z} - \frac{1}{\xi - z^*} \right)(\xi - z_0)}\, f(\xi)\, |d\xi|$$

Poniamo

armonica anti-armonica

$$P(z;\xi) = \frac{1}{2\pi R} \left(\frac{1}{\xi - z} - \frac{1}{\xi - z^*} \right)(\xi - z_0)$$

che si chiama

Nucleo di Poisson per il disco $B_R(z_0)$

$$\boxed{\frac{1}{\xi - z} - \frac{1}{\xi - z^*}} = \frac{1}{(\xi - z_0) - (z - z_0)} - \frac{1}{(\xi - z_0) - \frac{R^2}{\overline{(z - z_0)}}} =$$

$$= \frac{(z-z_0) - \frac{R^2}{\overline{(z-z_0)}}}{\left((\xi-z_0)-(z-z_0)\right)\left((\xi-z_0) - \frac{R^2}{\overline{z-z_0}}\right)} =$$

$$= \frac{\left(|z-z_0|^2 - R^2\right)\cancel{\frac{1}{\overline{z-z_0}}}}{\underbrace{\left((\xi-z_0)-(z-z_0)\right)}\left((\xi-z_0)\overline{(z-z_0)} - R^2\right)\cancel{\frac{1}{\overline{z-z_0}}}} =$$

$$R^2 = |\xi-z_0|^2 = (\xi-z_0)\overline{(\xi-z_0)}$$

$$= \frac{|z-z_0|^2 - R^2}{(\xi-z)\left((\xi-z_0)\overline{(z-z_0)} - (\xi-z_0)\overline{(\xi-z_0)}\right)} =$$

$$= \frac{|z-z_0|^2 - R^2}{(\xi-z)(\xi-z_0)\overline{(z-\xi)}} =$$

$$= \frac{R^2 - |z-z_0|^2}{(\xi-z_0)\,|\xi-z|^2}$$

$$\boxed{P(z;\xi) = \frac{1}{2\pi R}\,\frac{R^2-|z-z_0|^2}{|\xi-z|^2}}$$

$$f(z) = \frac{1}{2\pi R} \int_{\partial B_R(z_0)} \frac{R^2 - |z - z_0|^2}{|\xi - z|^2} f(\xi)\, |d\xi| . \qquad \square$$

$$f(z) = \int_{\partial B_R(z_0)} P(z;\xi)\, f(\xi)\, |d\xi|$$

Teor. Sia $u \in C^2(B_R(z_0)) \cap C(\overline{B_R(z_0)})$ armonica in $B_R(z_0)$. allora vale

$$u(z) = \int_{\partial B_R(z_0)} P(z;\xi)\, u(\xi)\, |d\xi| .$$

Dim. Sia $z \in B_R(z_0)$, sia $r > 0$ t.c.

$$|z - z_0| < r < R$$

Sia $f \in H(B_R(z_0))$ t.c. $u = \operatorname{Re} f$.

Sia P il nucleo di Poisson su $B_r(z_0)$

$$P(z;\xi) = \frac{1}{2\pi r} \; \frac{r^2 - |z - z_0|^2}{|\xi - z|^2}$$

vale $$f(z) = \int_{\partial B_r(z_0)} P(z;\xi)\, f(\xi)\, |d\xi|$$

separando la parte reale ad ambo i membri

$$u(z) = \int_{\partial B_r(z_0)} P(z;\xi)\, u(\xi)\, |d\xi| =$$

$$= \frac{1}{2\pi r} \int_0^{2\pi} \underbrace{\frac{r^2 - |z - z_0|^2}{|re^{i\vartheta} - (z - z_0)|^2}\, u(z_0 + re^{i\vartheta})}_{g(r,\vartheta)}\, r\, d\vartheta$$

Per $z \in B_r(z_0)$ fissato, g è unif continua per $r \sim R$

$$u(z) = \frac{1}{2\pi R} \int_0^{2\pi} \frac{R^2 - |z - z_0|^2}{|Re^{i\vartheta} - (z - z_0)|^2}\, u(z_0 + Re^{i\vartheta})\, R\, d\vartheta$$

$$u(z) = \frac{1}{2\pi R} \int_{\partial B_R(z_0)} \frac{R^2 - |z - z_0|^2}{|\xi - z|^2} u(\xi)\, |d\xi|$$

$\square$

Oss. Posto $P(z;\xi) = \frac{1}{2\pi R} \frac{R^2 - |z-z_0|^2}{|\xi - z|^2}$

vale $\int_{\partial B_R(z_0)} P(z;\xi) |d\xi| = 1$

$$P(z;\xi) = \frac{1}{2\pi R}\left(\frac{1}{\xi - z} - \frac{1}{\xi - z^*}\right)(\xi - z_0)$$

$z \to P(z;\xi)$ è armonica.

Presa comunque $\varphi \in C(\partial B_R(z_0))$

$$u(z) = \int_{\partial B_R(z_0)} P(z,\xi)\, \varphi(\xi)\, |d\xi|$$

u risulta armonica in $B_R(z_0)$

Domanda $\lim\limits_{z \to \xi \in \partial B_R(z_0)} u(z) = \varphi(\xi)$?