Exercises QFT II — 2018/2019

Problem Sheet 7

Problem 13: β function for $\lambda\phi^4$ theory

In $\lambda\phi^4$ theory, the renormalized 1PI four point function is

$$
\begin{aligned}
i\tilde{\Gamma}^{(4)}(s,t,u;\lambda_r,m_r,\mu) \quad = \quad & -i\lambda_r - i\frac{\lambda_r^2}{32\pi^2}\int_0^1 dx \left[\ln\left(\frac{m_r^2 - x(1-x)s}{4\pi\mu^2}\right)\right. \\
& \left. + \ln\left(\frac{m_r^2 - x(1-x)t}{4\pi\mu^2}\right) + \ln\left(\frac{m_r^2 - x(1-x)u}{4\pi\mu^2}\right)\right]
\end{aligned} \tag{1}
$$

where λ_r is the renormalized coupling constant.
Let us suppose that by performing a measurement, we find

$$
\tilde{\Gamma}^{(4)}(4m^2,0,0;\lambda_r(\mu),m_r(\mu),\mu) = -L_{\mathrm{exp}} \tag{2}
$$

where L_{exp} is a measured number, that does not depend on the arbitrary scale μ (this implies that λ_r must depend on μ).

1. By deriving (2) with respect to μ, compute (**at order** λ_r^2)
$$
\mu\frac{d\lambda_r}{d\,\mu}\,. \tag{3}
$$

2. Integrate the differential equation and find the explicit expression of $\lambda_r(\mu)$ as a function of μ. Use the initial condition $\lambda_r(\mu_0) = \lambda_0$.

3. Compute the position of the Landu pole in the μ space.

4. Compute the scale $\bar{\mu}$ at which the theory becomes strongly coupled (i.e. $\lambda_r(\bar{\mu}) \simeq 1$).

5. Compute
$$
\lim_{\mu\to 0}\lambda_r(\mu)\,. \tag{4}
$$
What is the value of $\beta(\lambda_r) = \mu\frac{d\lambda_r}{d\,\mu}$ in this limit? Is there an IR fixed point?

Problem 14: γ matrices

Recall that the γ matrices satisfy the anticommutation relations $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}\mathbf{1}$ and that the Lorentz generators in the spinorial representation are written as $\mathcal{S}^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu]$. Moreover, $\gamma^5 = i\,\gamma^0\,\gamma^1\,\gamma^2\,\gamma^3$.

1) Prove that $\mathcal{S}^{\mu\nu}$ satisfy the same commutation relations of the Lorentz algebra generators $J^{\mu\nu}$, i.e.

$$[J^{\mu\nu}\,,\, J^{\rho\sigma}] = i\,(\eta^{\nu\rho}J^{\mu\sigma} - \eta^{\mu\rho}J^{\nu\sigma} - \eta^{\nu\sigma}J^{\mu\rho} + \eta^{\mu\sigma}J^{\nu\rho})\ . \tag{5}$$

2) Work out the value of the following expressions:

$$\begin{aligned}
\left(\gamma^5\right)^p &= ? && \text{p positive integer}\\
\left[\gamma^5,\, \gamma^\mu \ldots\ldots \gamma^\rho\right] &= ? && \text{even or odd number of } \gamma^\lambda\\
\left[\gamma^5,\, \mathcal{S}^{\mu\nu}\right] &= ? &&\\
\left[\gamma^\rho,\, \mathcal{S}^{\mu\nu}\right] &= ? &&\\
\left[\gamma^\rho\gamma^5,\, \mathcal{S}^{\mu\nu}\right] &= ? &&
\end{aligned}$$

A Lorentz transformation can be expressed as $\Lambda = e^{i\omega_{\mu\nu}\,J^{\mu\nu}}$ in terms of the generators $J^{\mu\nu}$ of the corresponding Lie Algebra. This is represented on the Dirac spinors by the matrix

$$R_s(\Lambda) = e^{i\omega_{\mu\nu}\,\mathcal{S}^{\mu\nu}}.$$

3) Prove that, for infinitesimal Lorentz transformation ($\omega_{\mu\nu} \ll 1$), the inverse action $R_s(\Lambda)^{-1}$ can be expressed as

$$R_s(\Lambda)^{-1} = \gamma^0\,R_s(\Lambda)^\dagger\,\gamma^0\,.$$

Is the previous formula valid also for $\omega^{\mu\nu}$ not necessarily small?