# Problema di Dirichlet nel disco.

Presa $\varphi \in C(\partial B_R(z_0))$, trovare
$u \in C^2(B_R(z_0)) \cap C(\overline{B_R(z_0)})$ t.c.

$$\begin{cases} \Delta u = 0 & \text{in } B_R(z_0) \\ u = \varphi & \text{su } \partial B_R(z_0) \end{cases}$$

Teor. di Schwarz

Il pb. di Dirichlet ha una unica soluzione che è data da

$$u(z) = \int_{\partial B_R(z_0)} P(z;\zeta)\, \varphi(\zeta)\, |d\zeta|$$

Dim. Unicità: se $u_1, u_2$ sono due soluzioni del pb. di Dirichlet, posto $u = u_1 - u_2$, abbiamo

$u \in C^2(B_R(z_0)) \cap C(\overline{B_R(z_0)})$ e

vale

$$\Delta u = 0 \quad \text{in } B_R(z_0)$$
$$u = 0 \quad \text{su } \partial B_R(z_0)$$

$$u(z) = \int_{\partial B_R(z_0)} P(z;\xi) \cdot 0 \, |d\xi| \equiv 0 .$$

<u>Esistenza</u> Posto

$$\boxed{u(z) = \int_{\partial B_R(z_0)} P(z;\xi)\, \varphi(\xi)\, |d\xi|}$$

si tratta di verificare che $u$ è armonica in $B_R(z_0)$ e che preso $w \in \partial B_R(z_0)$

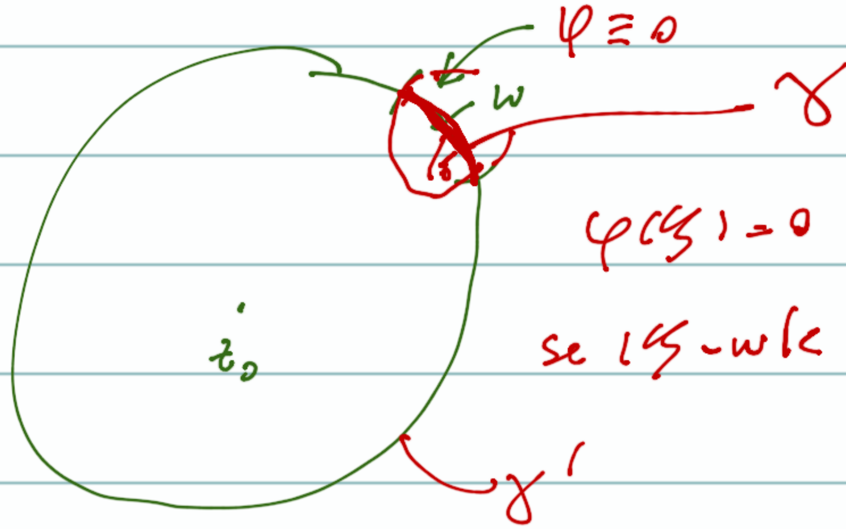
$\underline{u(z) \to \varphi(w)}$ se $z \to w$, $z \in B_R(z_0)$.

$z \to P(z;\xi)$ è armonica.

Derivando sotto $\int$ si ricava che anche $u$ è armonica nell'aperto.

$$\left[ P(z,\xi) = \frac{1}{2\pi R} \left( \underline{\frac{1}{\xi - z}} - \underline{\frac{1}{\xi - z^*}} \right) (\xi - z_0) \right]$$

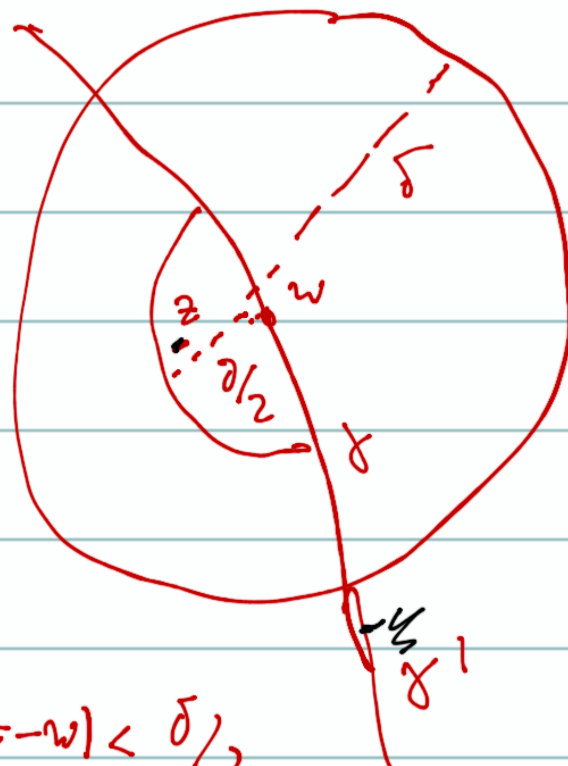
Supp. per il momento che $\varphi \equiv 0$ per $\xi$ vicino a $w$:




$\varphi(\xi) = 0$ se $|\xi - w| < \delta$ ||

Sia $\gamma = \partial B_R(z_0) \cap B_\delta(w)$

$\gamma' = \partial B_R(z_0) \setminus B_\delta(w)$




$|z-w| < \delta/2$

Se $\xi \in \gamma'$ $\quad |z-\xi| \geq \frac{\delta}{2}$

$$u(z) = \int_{\partial B_R(z_0)} P(z,\xi)\varphi(\xi)\,|d\xi| =$$

$$= \int_{\gamma'} \frac{1}{2\pi R} \frac{R^2 - |z-z_0|^2}{\underbrace{|\xi - z|^2}} \varphi(\xi)\, |d\xi| =$$

$|\xi - z| \geq \frac{\delta}{2}$

Sia $M$ t.c. $|\varphi| \leq M$ su $\partial B_R(z_0)$

$$|u(z)| \leq \underbrace{\int_{\gamma'} \frac{1}{2\pi R} \frac{R^2 - |z-z_0|^2}{\boxed{\left(\frac{\delta}{2}\right)^2}} M\, |d\xi|} \leq$$

$$\leq M \left(\frac{2}{\delta}\right)^2 (R^2 - \underbrace{|z-z_0|^2}_{\downarrow \atop R^2}) \xrightarrow[z \to w]{} 0$$

Quindi

$$u(z) \to 0 = \varphi(w) \quad \text{se } \underline{z \to w}.$$

<u>Caso</u> <u>generale</u>. $\varphi$ è continua su $\partial B_R(z_0)$: quindi $\forall \varepsilon > 0 \ \exists \delta > 0$

t.c. se $|\xi - w| < \delta$ :

$$|\varphi(\xi) - \varphi(w)| < \varepsilon$$

$$u(z) - \varphi(w) = \int_{\partial B_R(z_0)} P(z,\xi)\big(\varphi(\xi) - \varphi(w)\big)\,|d\xi|$$

Pongo $\gamma$, $\gamma'$ come lì.

$$u(z) - \varphi(w) = \int_{\gamma} '' + \int_{\gamma'} P(z,\xi)\big(\varphi - \varphi(w)\big)\,|d\xi|$$

$$|u(z) - \varphi(w)| \le \underbrace{\int_{\gamma} P\,|\varphi - \varphi(w)|} + \int_{\gamma'} P\,|\varphi - \varphi(w)|\,|d\xi|$$

$$\int_{\gamma} P(z,\xi)\,|\varphi(\xi) - \varphi(w)|\,|d\xi| \le$$

$$\le \varepsilon \int_{\gamma} P(z,\xi)\,|d\xi| \le \varepsilon \quad .$$

$$\int_{\gamma'} P(z,\xi)\, |\varphi(\xi) - \varphi(w)|\, |d\xi|$$

Posto $M = \max\limits_{\partial B_R(z_0)} |\varphi|$ ,

$$|\varphi(\xi) - \varphi(w)| \leq 2M$$

$$\int_{\gamma'} P(z,\xi) |\varphi(\xi) - \varphi(w)|\, |d\xi| \leq$$

$$\leq 2M \int_{\gamma'} P(z;\xi)\, |d\xi|$$

Ora se $|z-w| < \frac{\delta}{2}$, $|z-\xi| \geq \frac{\delta}{2}$ $\forall \xi \in \gamma'$

$$P(z,\xi) \leq \frac{1}{2\pi R} \frac{R^2 - |z-z_0|^2}{\left(\frac{\delta}{2}\right)^2} \xrightarrow[z \to w]{} 0$$

Quindi

$$\limsup_{z \to w} |u(z) - \varphi(w)| \leq \varepsilon \qquad \forall \varepsilon > 0$$

Ovvero

$$u(z) \to v(w) \quad \text{per } z \to w$$

$$z \in B_R(z_0).$$

$\square$

$$P(z;\xi) = \frac{1}{2\pi R} \frac{R^2 - \cancel{|z-z_0|^2}}{\underbrace{|\xi - z|^2}_{R^2}}$$

Sia $z = z_0$ ;

$$P(z_0;\xi) = \frac{1}{2\pi R}$$

$$u(z_0) = \frac{1}{2\pi R} \int_{\partial B_R(z_0)} u(\xi) \, |d\xi|$$

Teorema del valor medio.

Se $u \in C^2(B_R(z_0)) \cap C(\overline{B_R(z_0)})$ è armonica nel disco, allora:

$$u(z_0) = \frac{1}{2\pi R} \int_{\partial B_R(z_0)} u(\xi)\,|d\xi| \quad ,$$

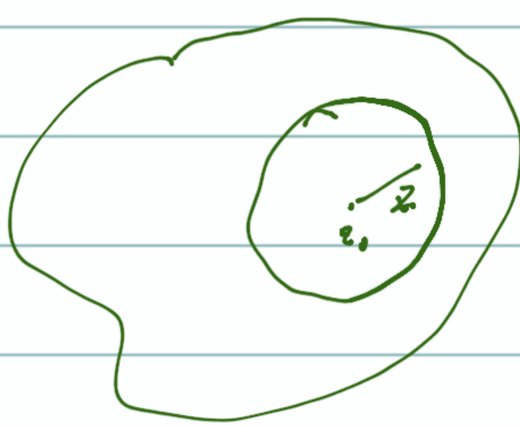
<u>Def.</u> Sia $\Omega$ aperto di $\mathbb{R}^2$. Sia $u \in C(\Omega; \mathbb{R})$. Si dice che $u$ ha la proh. del valor medio se per ogni $z \in \Omega$ $\exists \{r_n\}$, $r_n \searrow 0$ t.c.

$$B_{r_n}(z) \subset \Omega \quad \text{e}$$

$$u(z) = \frac{1}{2\pi r_n} \int_{\partial B_{r_n}(z)} u(\xi)\,|d\xi| \quad \forall z, \forall r$$

<u>Teorema</u> Se $u \in C(\Omega; \mathbb{R})$ ha la propr. del valor medio allora è armonica in $\Omega$.

<u>Dim.</u>



Sia $\overline{B_R(z_0)} \subset \Omega$

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Sia

$$h(z) = \int_{\partial B_R(z_0)} P(z, \xi)\, u(\xi)\, |d\xi|$$

$h$ verifica

$$\begin{cases} \Delta h = 0 & \text{in } B_R(z_0) \\ h = u & \text{su } \partial B_R(z_0) \end{cases}$$

Proviamo che $u \equiv h$ in $B_R(z_0)$.

Poniamo $v = u - h$, avremo

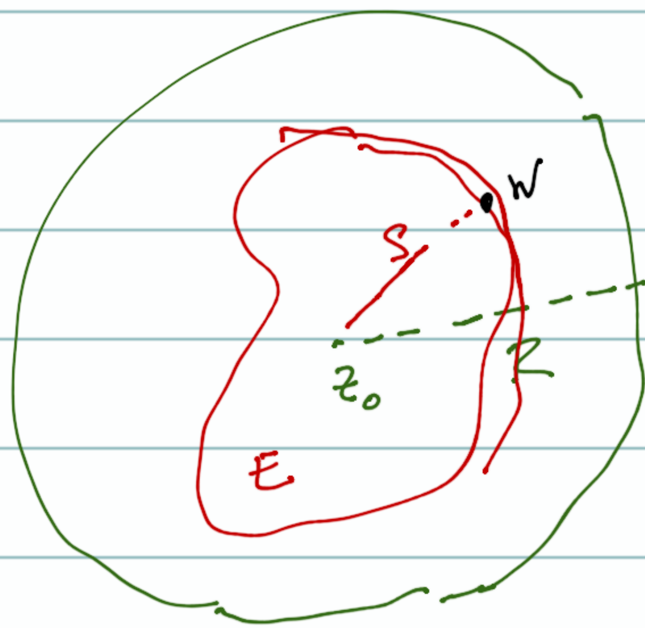
$$v = 0 \quad \text{su} \quad \partial B_R(z_0)$$

Supp. per assurdo che $v$ assume valori positivi in $B_R(z_0)$, poniamo

$$m = \max_{\overline{B_R(z_0)}} v > 0$$

Sia $E = \{ z \in B_R(z_0) \mid v(z) = m \}$.

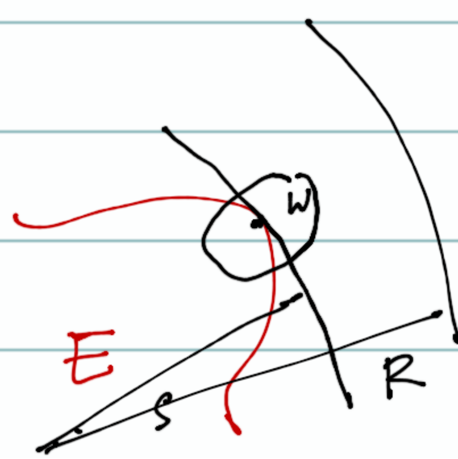
$E$ è un chiuso $\subset B_R(z_0)$, quindi compatto.




Sia $g(z) = |z - z_0|$ è funzione cont. su $E$

sia $\rho = \max_E |g| < R$, e sia

$w \in E$ t.c. $|w - z_0| = g(w) = \rho$

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Per ipotesi, $u$ verifica la prop. del v. medio

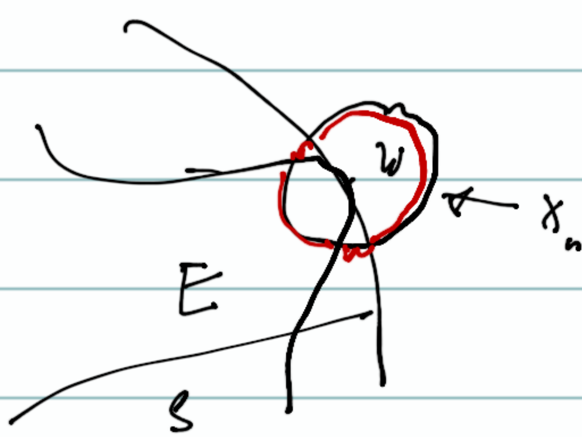
quindi esiste $\{r_n\}$, $r_n \to 0$, $B_{r_n}(w) \subset B_R(z_0)$

e $$u(w) = \frac{1}{2\pi r_n} \int_{\partial B_{r_n}(w)} u(\xi)\,|d\xi|$$

abbiamo anche
$$h(w) = \frac{1}{2\pi r_n} \int_{\partial B_{r_n}(w)} h(\xi)\,|d\xi|$$

sottraendo:

$$\underset{m\,!}{\underline{v(w)}} = \underbrace{\frac{1}{2\pi r_n} \int_{\underline{\partial B_{r_n}(w)}} \overset{\swarrow}{v(\xi)}\,|d\xi|}$$




$\gamma_n = \partial B_{r_n}(w) \setminus \overline{B_\rho(z_0)}$

$\gamma_n' = \partial B_{r_n}(w) \cap B_\rho(z_0)$

$|\gamma_n| \geq \pi r_n > 0$ e su $\gamma_n$

$v < m$

$$\left|\int_{\gamma_n} v\right| < \underline{\underline{m\, L(\gamma_n)}}$$

$$\left|\int_{\gamma_n'} v\right| \underset{\text{A}}{\leq} \underline{\underline{m\, L(\gamma_n')}}$$

$$\Downarrow$$

$$\int_{\partial B_{r_n}(w)} v\,|d\xi| < m\left(L(\gamma_n) + L(\gamma_n')\right) = m\, 2\pi r_n$$

$$v(w) \neq \frac{1}{2\pi r_n}\int_{\partial B_{r_n}(w)} v\,|d\xi| < \underline{\underline{m}}$$

Assurdo.

Quindi $v$ non può essere valori positivi in $B_R(z_0)$. In modo analogo si tira che non può assumere val. negativi.

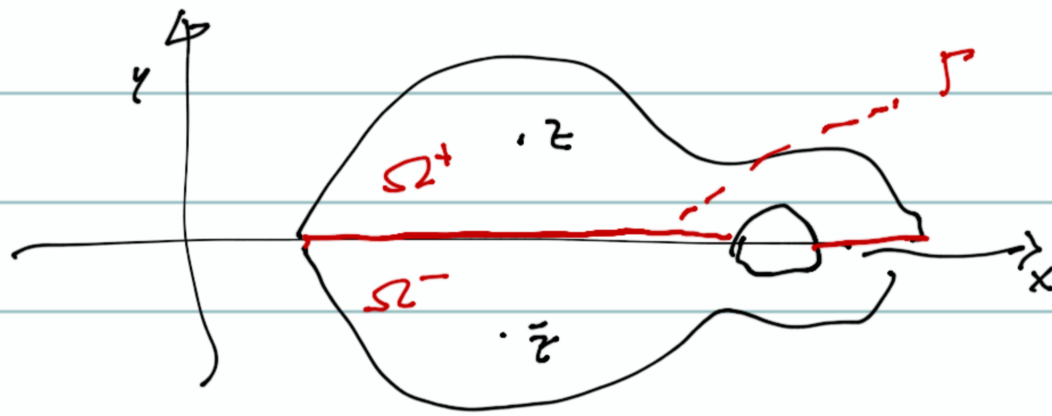
Di conseguenza

$$u = h \quad \text{in } B_R(z_0)$$

Per l'arbitrarietà di $B_R(z_0) \subset \Omega$

u risulta armonica in tutto $\Omega$. $\square$

Principio di riflessione di Schwarz.




Def. Un aperto $\Omega \subset \mathbb{C}$ si dice simmetrico rispetto all'asse reale se per ogni $z \in \Omega$ abbiamo che anche $\bar{z} \in \Omega$.

Poniamo $\Omega^+ = \{ z \in \Omega \mid \operatorname{Im} z > 0 \}$

$$\Omega^- = \{ z \in \Omega \mid \operatorname{Im} z < 0 \}$$

$$\Gamma = \{ z \in \Omega \mid \operatorname{Im} z = 0 \}$$

<u>Teor.</u> Sia $u$ armonica in $\Omega^+$ t.c.

$\forall z_0 \in \Gamma$

$$\lim_{\substack{z \to z_0 \\ z \in \Omega^+}} u(z) = 0$$

Allora posto

$$w(z) = \begin{cases} u(z) & z \in \Omega^+ \\ 0 & z \in \Gamma \\ -u(\bar{z}) & z \in \Omega^- \end{cases}$$

$w$ è armonica in tutto $\Omega$.