10. Products of quasi–projective varieties, tensors and Grassmannians.

a) Products

Let $\mathbb{P}^n$, $\mathbb{P}^m$ be projective spaces over the same field K. The cartesian product $\mathbb{P}^n \times \mathbb{P}^m$ is simply a set: we want to define an injective map from $\mathbb{P}^n \times \mathbb{P}^m$ to a suitable projective space, so that the image is a projective variety, which will be identified with our product.

Let $N = (n+1)(m+1)-1$ and define $\sigma : \mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^N$ in the following way: $\sigma([x_0, \dots, x_n], [y_0, \dots, y_m]) = [x_0y_0, x_0y_1, \dots, x_iy_j, \dots, x_ny_m]$. Using coordinates w_{ij}, $i = 0, \dots, n$, $j = 0, \dots, m$, in $\mathbb{P}^N$, σ is defined by

$$\{w_{ij} = x_iy_j, i = 0, \dots, n, j = 0, \dots, m.$$

It is easy to observe that σ is a well–defined map.

Let $\Sigma_{n,m}$ (or simply Σ) denote the image $\sigma(\mathbb{P}^n \times \mathbb{P}^m)$.

10.1. Proposition. *σ is injective and $\Sigma_{n,m}$ is a closed subset of $\mathbb{P}^N$.*

Proof. If $\sigma([x], [y]) = \sigma([x'], [y'])$, then there exists $\lambda \neq 0$ such that $x'_iy'_j = \lambda x_iy_j$ for all i, j. In particular, if $x_h \neq 0$, $y_k \neq 0$, then also $x'_h \neq 0$, $y'_k \neq 0$, and for all i $x'_i = \lambda \frac{y_k}{y'_k} x_i$, so $[x_0, \dots, x_n] = [x'_0, \dots, x'_n]$. Similarly for the second point.
To prove the second assertion, I claim: $\Sigma_{n,m}$ is the closed set of equations:

$$(*)\{w_{ij}w_{hk} = w_{ik}w_{hj},\ i, h = 0, \dots, n; j, k = 0 \dots, m.$$

It is clear that if $[w_{ij}] \in \Sigma$, then it satisfies (*). Conversely, assume that $[w_{ij}]$ satisfies (*) and that $w_{\alpha\beta} \neq 0$. Then

$$\begin{aligned}[w_{00}, \dots, w_{ij}, \dots, w_{nm}] &= [w_{00}w_{\alpha\beta}, \dots, w_{ij}w_{\alpha\beta}, \dots, w_{nm}w_{\alpha\beta}] = \\ &= [w_{0\beta}w_{\alpha 0}, \dots, w_{i\beta}w_{\alpha j}, \dots, w_{n\beta}w_{\alpha m}] = \\ &= \sigma([w_{0\beta}, \dots, w_{n\beta}], [w_{\alpha 0}, \dots, w_{\alpha m}]).\end{aligned}$$

□

σ is called the Segre map and $\Sigma_{n,m}$ the Segre variety or biprojective space. Note that Σ is covered by the affine open subsets $\Sigma^{ij} = \Sigma \cap W_{ij}$, where $W_{ij} = \mathbb{P}^N \setminus V_P(w_{ij})$. Moreover $\Sigma^{ij} = \sigma(U_i \times V_j)$, where $U_i \times V_j$ is naturally identified with $\mathbb{A}^{n+m}$.

10.2. Proposition. *$\sigma|_{U_i \times V_j} : U_i \times V_j = \mathbb{A}^{n+m} \to \Sigma^{ij}$ is an isomorphism of varieties.*

Proof. Assume by simplicity $i = j = 0$. Choose non–homogeneous coordinates on U_0: $u_i = x_i/x_0$ and on V_0: $v_j = y_j/y_0$. So $u_1, \dots u_n, v_1, \dots, v_m$ are coordinates on

$U_0 \times V_0$. Take non–homogeneous coordinates also on W_{00}: $z_{ij} = w_{ij}/w_{00}$. Using these coordinates we have:

$$\sigma|_{U_i\times V_j} :(u_1,\dots u_n, v_1,\dots,v_m) \to (v_1,\dots,v_m,u_1,u_1v_1,\dots,u_1v_m,\dots,u_nv_m)$$
$$||$$
$$([1,u_1,\dots,u_n],[1,v_1,\dots,v_m])$$

i.e. $\sigma(u_1,\dots,v_m) = (z_{01},\dots,z_{nm})$, where

$$\begin{cases} z_{i0} = u_i, & \text{if } i = 1,\dots,n; \\ z_{0j} = v_j, & \text{if } j = 1,\dots,m; \\ z_{ij} = u_iv_j = z_{i0}z_{0j} & \text{otherwise.} \end{cases}$$

Hence $\sigma|_{U_0\times V_0}$ is regular.
The inverse map takes $(z_{01},\dots,z_{nm})$ to $(z_{10},\dots,z_{n0},z_{01},\dots,z_{0m})$, so it is also regular. □

10.3. Corollary. *$\mathbb{P}^n \times \mathbb{P}^m$ is irreducible and birational to $\mathbb{P}^{n+m}$.*

Proof. The first assertion follows from Ex.5, Ch.6, considering the covering of Σ by the open subsets Σ^{ij}. Indeed, $\Sigma^{ij} \cap \Sigma^{hk} = \sigma((U_i \times V_j) \cap (U_h \times V_k)) = \sigma((U_i \cap U_h) \times (V_j \cap V_k))$, and $U_i \cap U_h \neq \emptyset \neq V_j \cap V_k$.

For the second assertion, by Theorem 9.15, it is enough to note that $\Sigma_{n,m}$ and $\mathbb{P}^{n+m}$ contain isomorphic open subsets, i.e. Σ^{ij} and $\mathbb{A}^{n+m}$. □

From now on, we shall identify $\mathbb{P}^n \times \mathbb{P}^m$ with $\Sigma_{n,m}$. If $X \subset \mathbb{P}^n$, $Y \subset \mathbb{P}^m$ are any quasi–projective varieties, then $X \times Y$ will be automatically identified with $\sigma(X \times Y) \subset \Sigma$.

10.4. Proposition. *If X and Y are projective varieties (resp. quasi–projective varieties), then $X \times Y$ is projective (resp. quasi–projective).*

Proof.

$$\begin{aligned}\sigma(X\times Y) &= \bigcup_{i,j}(\sigma(X\times Y)\cap\Sigma^{ij}) = \\ &= \bigcup_{i,j}(\sigma(X\times Y)\cap(U_i\times V_j)) = \\ &= \bigcup_{i,j}(\sigma((X\cap U_i)\times(Y\cap V_j))).\end{aligned}$$

If X and Y are projective varieties, then $X \cap U_i$ is closed in U_i and $Y \cap V_j$ is closed in V_j, so their product is closed in $U_i \times V_j$; since $\sigma|_{U_i\times V_j}$ is an isomorphism, also $\sigma(X \times Y) \cap \Sigma^{ij}$ is closed in Σ^{ij}, so $\sigma(X \times Y)$ is closed in Σ, by Lemma 8.3.

If X, Y are quasi–projective, the proof is similar: $X \cap U_i$ is locally closed in U_i and $Y \cap V_j$ is locally closed in V_j, so $X \cap U_i = Z \setminus Z'$, $Y \cap V_j = W \setminus W'$, with

Z, Z', W, W' closed. Therefore $(Z \setminus Z') \times (W \setminus W') = Z \times W \setminus ((Z' \times W) \cup (Z \times W'))$, which is locally closed.

As for the irreducibility, see Exercise 10.1. □

10.5. Example. $\mathbb{P}^1 \times \mathbb{P}^1$
$\sigma : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^3$ is given by $\{w_{ij} = x_i y_j,\ i = 0, 1,\ j = 0, 1.$ Σ has only one non–trivial equation: $w_{00}w_{11} - w_{01}w_{10}$, hence Σ is a quadric. The equation of Σ can be written as

$$(*) \qquad \begin{vmatrix} w_{00} & w_{01} \\ w_{10} & w_{11} \end{vmatrix} = 0.$$

Σ contains two families of special closed subsets parametrised by $\mathbb{P}^1$, i.e.

$$\{\sigma(P \times \mathbb{P}^1)\}_{P \in \mathbb{P}^1} \quad \text{and} \quad \{\sigma(\mathbb{P}^1 \times Q)\}_{Q \in \mathbb{P}^1}.$$

If $P[a_0, a_1]$, then $\sigma(P \times \mathbb{P}^1)$ is given by the equations:

$$\begin{cases} w_{00} = a_0 y_0 \\ w_{01} = a_0 y_1 \\ w_{10} = a_1 y_0 \\ w_{11} = a_1 y_1 \end{cases}$$

hence it is a line. Cartesian equations of $\sigma(P \times \mathbb{P}^1)$ are:

$$\begin{cases} a_1 w_{00} - a_0 w_{10} = 0 \\ a_1 w_{01} - a_0 w_{11} = 0; \end{cases}$$

they express the proportionality of the rows of the matrix (*) with coefficients $[a_1, -a_0]$. Similarly, $\sigma(\mathbb{P}^1 \times Q)$ is the line of equations

$$\begin{cases} a_1 w_{00} - a_0 w_{01} = 0 \\ a_1 w_{10} - a_0 w_{11} = 0. \end{cases}$$

Hence Σ contains two families of lines, called the rulings of Σ: two lines of the same ruling are clearly disjoint while two lines of different rulings intersect at one point $(\sigma(P, Q))$. Conversely, through any point of Σ there pass two lines, one for each ruling. Note that Σ is exactly the quadric surface of Example 9.17, d) and that the projection of centre $[1, 0, 0, 0]$ realizes an explicit birational map between $\mathbb{P}^1 \times \mathbb{P}^1$ and $\mathbb{P}^2$.

b) Tensors

The product of projective spaces has a coordinate-free description in terms of tensors. Precisely, let $\mathbb{P}^n = \mathbb{P}(V)$ and $\mathbb{P}^m = \mathbb{P}(W)$. The tensor product $V \otimes W$ of the vector spaces V, W is constructed as follows: let $K(V \times W)$ be the K-vector space with basis $V \times W$ obtained as the set of formal finite linear combinations of type $\Sigma_i a_i(v_i, w_i)$ with $a_i \in K$. Let U be the vector subspace generated by all elements of the form:

$(v, w) + (v', w) - (v + v', w)$,
$(v, w) + (v, w') - (v, w + w')$,
$(\lambda v, w) - \lambda(v, w)$,
$(v, \lambda w) - (\lambda(v, w)$,

with $v, v' \in V$, $w, w' \in W$, $\lambda \in K$. The tensor product is by definition the quotient $V \otimes W := K(V \times W)/U$. The class of a pair (v, w) is denoted $v \otimes w$, and called a decomposable tensor. So $V \otimes W$ is generated by the decomposable tensors; more precisely, a general element $\omega \in V \otimes W$ is of the form $\Sigma_{i=1}^k v_i \otimes w_i$, with $v_i \in V$, $w_i \in W$. The minimum k such that an expression of this type exists is called the tensor rank of ω.

There is a natural bilinear map $\otimes : V \times W \to V \otimes W$, such that $(v, w) \to v \otimes w$. It enjoys the following universal property: for any K-vector space Z with a bilinear map $f : V \times W \to Z$, there exists a unique linear map $\bar{f} : V \otimes W \to Z$ such that f factorizes in the form $f = \bar{f} \circ \otimes$.

If $\dim V = n$, $\dim W = m$, and bases $\mathcal{B} = (e_1, \dots, e_n), \mathcal{B}' = (e'_1, \dots, e'_m)$ are given, then $(e_1 \otimes e'_1, \dots, e_i \otimes e'_j, \dots e_n \otimes e'_m)$ is a basis of $V \otimes W$: therefore $\dim V \otimes W = nm$.

If $v = x_1 e_1 + \dots x_n e_n$, $w = y_1 e'_1 + \dots y_m e'_m$, then $v \otimes w = \Sigma x_i y_j e_i \otimes e'_j$. So, passing to the projective spaces, the map $\otimes$ defines precisely the Segre map $\sigma : \mathbb{P}(V) \times \mathbb{P}(W) \to \mathbb{P}(V \otimes W)$, $([v], [w]) \to [v \otimes w]$. Indeed in coordinates we have $([x_0, \dots, x_n], [y_0, \dots, y_m]) \to [w_{00}, \dots, w_{nm}]$, with $w_{ij} = x_i y_j$. The image of $\otimes$ is the set of decomposable tensors, or rank one tensors.

The tensor product $V \otimes W$ has the same dimension, and is therefore isomorphic to the vector space of $n \times m$ matrices. The coordinates w_{ij} can be interpreted as the entries of such a $n \times m$ matrix. The equations of the Segre variety $\Sigma_{n,m}$ are the 2×2 minors of the matrix, therefore $\Sigma_{n,m}$ can be interpreted as the set of matrices of rank one.

The construction of the tensor product can be iterated, to construct $V_1 \otimes V_2 \otimes \dots \otimes V_r$. The following properties can easily be proved:

1. $V_1 \otimes (V_2 \otimes V_3) \simeq (V_1 \otimes V_2) \otimes V_3$;
2. $V \otimes W \simeq W \otimes V$;
3. $V^* \otimes W \simeq Hom(V, W)$, where $f \otimes w \to (V \to W : v \to f(v)w)$.

Also the Veronese morphism has a coordinate free description, in terms of symmetric tensors. Given a vector space V, for any $d \geq 0$ the d-th symmetric power of V, $S^d V$ or $Sym^d V$, is constructed as follows. We consider the tensor product of d copies of V: $V \otimes \dots \otimes V = V^{\otimes d}$, and we consider its subvector space U generated by tensors of the form $v_1 \otimes \dots v_d - v_{\sigma(1)} \otimes \dots \otimes v_{\sigma(d)}$, where σ varies in the symmetric group on d elements $\mathcal{S}_d$. Then by definition $S^d V := V^{\otimes d}/U$.

The equivalence class $[v_1 \otimes \ldots \otimes v_d]$ is denoted as a product $v_1 \ldots v_d$.

There is a natural multilinear and symmetric map $V \times \ldots \times V = V^d \to S^dV$, such that $(v_1, \ldots, v_d) \to v_1 \ldots v_d$, which enjoys the universal property. S^dV is generated by the products $v_1 \ldots v_d$.

S^dV can also be interpreted as a subspace of $V^{\otimes d}$, by considering the following map, that is an isomorphism to the image:

$$S^dV \to V^{\otimes d}, \quad v_1 \ldots v_d \to \Sigma_{\sigma \in \mathcal{S}_d} \frac{1}{d!} v_{\sigma(1)} \otimes \ldots \otimes v_{\sigma(d)}.$$

If $\mathcal{B} = (e_1, \ldots, e_n)$ is a basis of V, then it is easy to check that a basis of S^dV is formed by the monomials of degree d in $e_1, \ldots, e_n$; therefore $\dim S^dV = \binom{n+d-1}{d}$.

For instance, in S^2V the product v_1v_2 can be identified with $\frac{1}{2}(v_1 \otimes v_2 + v_2 \otimes v_1)$.

The symmetric algebra of V is $SV := \oplus_{d \geq 0} S^dV = K \oplus V \oplus S^2V \oplus \ldots$. An inner product can be naturally defined to give it the structure of a K-algebra, which results to be isomorphic to the polynomial ring in n variables, where $n = \dim V$.

If $char K = 0$ the Veronese morphism can be interpreted in the following way:

$$v_{n,d} : \mathbb{P}(V) \to \mathbb{P}(S^dV), \; [v] = [x_0e_0 + \ldots x_ne_n] \to [v^d] = [(x_0e_0 + \ldots + x_ne_n)^d].$$

Moreover S^2V can be interpreted as space of the symmetric $d \times d$ matrices, and the Veronese variety $V_{n,2}$ as the subset of the symmetric matrices of rank one.

In a similar way it is possible to define the exterior powers of the vector space V. One defines the d-th exterior power $\wedge^d V$ as the quotient $V^{\otimes d}/\Lambda$, where Λ is generated by the tensors of the form $v_1 \otimes \ldots \otimes v_i \otimes \ldots \otimes v_j \otimes \ldots \otimes v_d$, with $v_i = v_j$ for some $i \neq j$. The following notation is used: $[v_1 \otimes \ldots \otimes v_d] = v_1 \wedge \ldots \wedge v_d$.

There is a natural multilinear alternating map $V \times \ldots \times V = V^d \to \wedge^d V$, that enjoys the universal property. Given a basis of V as before, a basis of $\wedge^d V$ is formed by the tensors $e_{i_1} \wedge \ldots \wedge e_{i_d}$, with $1 \leq i_1 < \ldots < i_d \leq n$. Therefore $\dim \wedge^d V = \binom{n}{d}$. The exterior algebra of V is the following direct sum: $\wedge V = \oplus_{d \geq 0} \wedge^d V = K \oplus V \oplus \wedge^2 V \oplus \ldots$. To define an inner product that gives it the structure of algebra we can proceed as follows.

Step 1. Fixed $v_1, \ldots, v_q \in V$, define $f : V^d \to \wedge^{d+p} V$ posing $f(x_1, \ldots, x_d) = x_1 \wedge \ldots \wedge x_d \wedge v_1 \wedge \ldots \wedge v_q$. Since f results to be multilinear and alternating, by the universal property we get a factorization of f through $\wedge^d V$, which gives a linear map $\bar{f} : \wedge^d V \to \wedge^{d+p} V$, extending f. For any $\omega \in \wedge^d V$, we denote $\bar{f}(\omega)$ by $\omega \wedge v_1 \wedge \ldots \wedge v_d$.

Step 2. Fixed $\omega \in \wedge^d V$, consider the map $g : V^p \to \wedge^{d+p} V$ such that $g(y_1, \ldots, y_p) = \omega \wedge y_1 \wedge \ldots \wedge y_p$: it is multilinear and alternating, therefore it factorizes through $\wedge^p V$ and we get a linear map $\bar{g} : \wedge^p V \to \wedge^{d+p} V$, extending g. We denote $\bar{g}(\sigma) := \omega \wedge \sigma$.

Step 3. For any $d, p \geq 0$ we have got a map $\wedge : \wedge^d V \times \wedge^p V \to \wedge^{d+p} V$, that results to be bilinear, and extends to an inner product $\wedge : (\wedge V) \times (\wedge V) \to \wedge V$, which gives $\wedge V$ the required structure of algebra.

10.6. Proposition. *Let V be a vector space of dimension n.*

(i) Vectors $v_1, \ldots, v_p \in V$ are linearly dependent if and only if $v_1 \wedge \ldots \wedge v_p = 0$.

(ii) Let $v \in V$ be a non-zero vector, and $\omega \in \wedge^p V$. Then $\omega \wedge v = 0$ if and only if there exists $\Phi \in \wedge^{p-1} V$ such that $\omega = \Phi \wedge v$. In this case we say that v divides ω.

Proof. The proof of (i) is standard. If $\omega = \Phi \wedge v$, then $\omega \wedge v = (\Phi \wedge v) \wedge v = \Phi \wedge (v \wedge v) = 0$. Conversely, if $\omega \wedge v = 0$, $v \neq 0$, we choose a basis of V, $\mathcal{B} = (e_1, \ldots, e_n)$ with $e_1 = v$. Write $\omega = \Sigma_{i_1 < \ldots < i_p} a_{i_1 \ldots i_p} e_{i_1} \wedge \ldots \wedge e_{i_p}$. Then $0 = \omega \wedge e_1 = \Sigma_{i_1 < \ldots < i_p} (+-) a_{i_1 \ldots i_p} e_1 \wedge e_{i_1} \wedge \ldots \wedge e_{i_p}$. If $i_1 = 1$, the corresponding summand does not appear in this sum, so it remains a linear combination of linearly independent tensors, which implies $a_{i_1 \ldots i_p} = 0$ every time $i_1 > 1$. Therefore $\omega = e_1 \wedge \Phi$ for a suitable Φ.

□

10.7. Proposition. *Let $\omega \neq 0$ be an element of $\wedge^p V$. Then ω is totally decomposable if and only if the subspace of V: $W = \{v \in V \mid v$ divides $\omega\}$ has dimension p.*

Proof. If $\omega = x_1 \wedge \ldots \wedge x_p \neq 0$, then $x_1, \ldots, x_p$ are linearly independent and belong to W. So we can extend them to a basis of V adding vectors $x_{p+1}, \ldots, x_n$. If $v \in W$, $v = \alpha_1 x_1 + \ldots + \alpha_n x_n$, and v divides ω, then $\omega \wedge v = 0$, i.e. $x_1 \wedge \ldots \wedge x_n \wedge (\alpha_1 x_1 + \ldots + \alpha_n x_n) = 0$. This implies $\alpha_{p+1} x_1 \wedge \ldots \wedge x_p \wedge x_{p+1} + \ldots + \alpha_n x_1 \wedge \ldots \wedge x_p \wedge x_n$, therefore $\alpha_{p+1} = \ldots = \alpha_n = 0$, so $v \in \langle x_1, \ldots, x_n \rangle$.

Conversely, if $(x_1, \ldots, x_p)$ is a basis of W, we can complete it to a basis of V and write $\omega = \Sigma a_{i_1 \ldots i_p} x_{i_1} \wedge \ldots \wedge x_{i_p}$. But x_1 divides ω, so $\omega \wedge x_1 = 0$. Replacing ω with its explicit expression, we obtain that $a_{i_1 \ldots i_p} = 0$ if $1 \notin \{i_1, \ldots, i_p\}$. Repeating this argument for $x_2, \ldots, x_p$, it remains $\omega = a_{1 \ldots p} x_1 \wedge \ldots \wedge x_p$. □

With explicit computations, one can prove the following proposition.

10.8. Proposition. *Let V be a vector space with $\dim V = n$. Let $\mathcal{B} = (e_1, \ldots, e_n)$ be a basis of V and $v_1, \ldots, v_n$ be any vectors. Then $v_1 \wedge \ldots \wedge v_n = \det(A) e_1 \wedge \ldots \wedge e_n$, where A is the matrix of the coordinates of the vectors $v_1, \ldots, v_n$ with respect to $\mathcal{B}$.*

10.9. Corollary. *Let $v_1, \ldots, v_p \in V$, with $v_j = \Sigma a_{ij} e_j$, $j = 1, \ldots, p$. Then $v_1 \wedge \ldots \wedge v_p = \Sigma_{i_1 < \ldots < i_p} a_{i_1 \ldots i_p} e_{i_1} \wedge \ldots \wedge e_{i_p}$, with $a_{i_1 \ldots i_p} = \det(A_{i_1 \ldots i_p})$, the determinant of the $p \times p$ submatrix of A containing the columns of indices $i_1, \ldots, i_p$.*

c) Grassmannians

Let V be a vector space of dimension n, and r be a positive integer, $1 \leq r \leq n$. The Grassmannian $G(r,V)$ is the set of the subspaces of V of dimension r. It can be denoted also $G(r,n)$.

There is a natural bijection between $G(r,V)$ and the set of the projective subspaces of $\mathbb{P}(V)$ of dimension $r-1$, denoted $\mathbb{G}(r-1,\mathbb{P}(V))$ or $\mathbb{G}(r-1,n-1)$. Let $W \in G(r,V)$; if $(w_1,\dots,w_r)$ and $(x_1,\dots,x_r)$ are two bases of W, then $w_1 \wedge \dots \wedge w_r = \lambda x_1 \wedge x_r$, where $\lambda \in K$ is the determinant of the matrix of the change of basis. Therefore W uniquely determines an element of $\wedge^r V$ up to proportionality. This allows to define a map, called the Plücker map, $\psi : G(r,V) \to \mathbb{P}(\wedge^r V)$, such that $\psi(W) = [w_1 \wedge \dots w_r]$.

10.10. Proposition. *The Plücker map is injective.*

Proof. Assume $\psi(W) = \psi(W')$, where W, W' are subspaces of V of dimension r with bases $(x_1,\dots,x_r)$ and $(y_1,\dots,y_r)$. So there exists $\lambda \neq 0$ in K such that $x_1 \wedge \dots \wedge x_r = \lambda y_1 \wedge \dots \wedge y_r$. This implies $x_1 \wedge \dots \wedge x_r \wedge y_i = 0$ for all i, so y_i is linearly dependent from $x_1,\dots,x_r$, so $y_i \in W$. Therefore $W' \subset W$. The reverse inclusion is similar. □

In coordinates, $\psi(W)$ is given by the minors of maximal order r of the matrix of the coordinates of the vectors of a basis of W, with respect to a fixed basis of V.

10.11. Examples.

(i) $r = n-1$: $\wedge^{n-1}V$ has dimension n, so it is isomorphic to the dual vector space V^*, associating to $e_1 \wedge \dots \wedge \hat{e}_k \wedge \dots \wedge e_n$ the linear form e_k^* of the dual basis. In this case the Plücker map is surjective, so $G(n-1,n) \simeq V^*$.

(ii) $n = 4, r = 2$: $G(2,4)$ or $\mathbb{G}(1,3)$, the Grassmannian of lines of $\mathbb{P}^3$. In this case $\psi : \mathbb{G}(1,3) \to \mathbb{P}(\wedge^2 V) \simeq \mathbb{P}^5$. Let (e_0,e_1,e_2,e_3) be a basis of V. If ℓ is the line of $\mathbb{P}^3$ obtained by projectivisation of a subspace $L \subset V$ of dimension 2, let $L = \langle x,y \rangle$; then $\psi(\ell) = [x \wedge y]$. Its Plücker coordinates are denoted $p_{01}, p_{02}, p_{03}, p_{12}, p_{13}, p_{23}$, and $p_{ij} = x_i y_j - x_j y_i$, the 2×2 minors of the matrix

$$\begin{pmatrix} x_0 & x_1 & x_2 & x_3 \\ y_0 & y_1 & y_2 & y_3 \end{pmatrix}.$$

This time ψ is not surjective; its image is formed by the totally decomposable tensors. They satisfy the equation of degree 2: $p_{01}p_{23} - p_{02}p_{13} + p_{03}p_{12} = 0$, which represents a quadric of maximal rank in $\mathbb{P}^5$, called the Klein quadric. The fact that this equation is satisfied can be seen by considering the 4×4 matrix

$$\begin{pmatrix} x_0 & x_1 & x_2 & x_3 \\ y_0 & y_1 & y_2 & y_3 \\ x_0 & x_1 & x_2 & x_3 \\ y_0 & y_1 & y_2 & y_3 \end{pmatrix}:$$

its determinant is precisely the above equation.

For instance the line of equations $x_2 = x_3 = 0$, obtained projectivising the subspace $\langle e_0, e_1\rangle$, has Plücker coordinates $[1, 0, 0, 0, 0, 0]$.

In general we can prove the following theorem.

10.12. Theorem. *The image of the Plücker map is a closed subset in* $\mathbb{P}(\wedge^r V)$.

Proof. The image of the Plücker map is the set of the proportionality classes of totally decomposable tensors. By Proposition 10.7, a tensor $\omega \in \wedge^r V$ is totally decomposable if and only if the subspace $W = \{v \in V \mid v \text{ divides } \omega\}$ has dimension r. We consider the linear map $\Phi : V \to \wedge^{r+1} V$, such that $\Phi(v) = \omega \wedge v$. The kernel of Φ is equal to W. So ω is totally decomposable if and only if the rank of Φ is $n-r$. Fixed a basis $\mathcal{B} = (e_1, \dots, e_n)$ of V, we write $\omega = \Sigma_{i_1<\dots<i_r} a_{i_1\dots i_r} e_{i_1} \wedge \dots \wedge e_{i_r}$. We then consider the basis of $\wedge^{r+1} V$ associated to $\mathcal{B}$ and we construct the matrix A of Φ with respect to these bases: its minors of order $n - p + 1$ are equations of the image of ψ, and they are polynomials in the coordinates $a_{i_1\dots i_r}$ of ω. □

From now on we shall identify the Grassmannian with the projective algebraic set that is its image in the Plücker map. The equations obtained in Theorem 10.12 are nevertheless not generators for the ideal of the Grassmannian. For instance, in the case $n = 4, r = 2$, let $\omega = p_{01} e_0 \wedge e_1 + p_{02} e_0 \wedge e_2 + \dots$. Then:

$\Phi(e_0) = \omega \wedge e_0 = p_{12} e_0 \wedge e_1 \wedge e_2 + p_{13} e_0 \wedge e_1 \wedge e_3 + p_{23} e_0 \wedge e_2 \wedge e_3;$

$\Phi(e_1) = \omega \wedge e_1 = -p_{02} e_0 \wedge e_1 \wedge e_2 - p_{03} e_0 \wedge e_1 \wedge e_3 + p_{23} e_1 \wedge e_2 \wedge e_3;$

$\Phi(e_2) = \omega \wedge e_2 = p_{01} e_0 \wedge e_1 \wedge e_2 - p_{03} e_0 \wedge e_2 \wedge e_3 + p_{13} e_1 \wedge e_2 \wedge e_3;$

$\Phi(e_3) = \omega \wedge e_3 = p_{01} e_0 \wedge e_1 \wedge e_3 + p_{02} e_0 \wedge e_2 \wedge e_3 + p_{12} e_1 \wedge e_2 \wedge e_3.$

So the matrix is

$$\begin{pmatrix} p_{12} & -p_{02} & p_{01} & 0 \\ p_{13} & -p_{03} & 0 & p_{01} \\ p_{23} & 0 & -p_{03} & p_{02} \\ 0 & p_{23} & p_{13} & p_{12} \end{pmatrix}.$$

Its 3×3 minors are equations defining $\mathbb{G}(1, 3)$, but the radical of the ideal generated by these minors is in fact $(p_{01}p_{23} - p_{02}p_{13} + p_{03}p_{12})$.

To find equations for the Grassmannian and to prove that it is irreducible, it is convenient to give an explicit open covering with affine open subsets. In $\mathbb{P}(\wedge^r V)$, let $U_{i_1\dots i_r}$ be the affine open subset where the Plücker coordinate $p_{i_1\dots i_r} \neq 0$. For semplicity assume $i_1 = 1, i_2 = 2, \dots, i_r = r$, and put $U = U_{1\dots r}$. If $W \in G(r, n) \cap U$, and $w_1, \dots, w_r$ is a basis of W, then the first minor of the matrix M, of the coordinates of $w_1, \dots, w_r$ with respect to a fixed basis of V, is non-degenerate. So we can choose a new basis of W such that M is of the form

$$M = \begin{pmatrix} 1 & 0 & \dots & 0 & \alpha_{1,r+1} & \dots & \alpha_{1,n} \\ 0 & 1 & \dots & 0 & \alpha_{2,r+1} & \dots & \alpha_{2,n} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 & \alpha_{r,r+1} & \dots & \alpha_{r,n} \end{pmatrix}.$$

Conversely, any matrix of this form defines a subspace $W \in G(r,n) \cap U$. So there is a bijection between $G(r,n) \cap U$ and $K^{r(n-r)}$, i.e. the affine space of dimension $r(n-r)$. The coordinates of W result to be equal to 1 and all minors of all orders of the submatrix of the last $n-r$ columns of M. Therefore they are expressed as polynomials in the $r(n-r)$ coordinates elements of M. This shows that $G(r,n) \cap U$ is an affine rational subvariety of U. By homogenising the equations obtained in this way, one gets equations for $G(r,n)$.

In the case $n = 4, r = 2$, the matrix M becomes

$$M = \begin{pmatrix} 1 & 0 & \alpha_{13} & \alpha_{14} \\ 0 & 1 & \alpha_{23} & \alpha_{24} \end{pmatrix}.$$

One gets $1 = p_{01}, \alpha_{23} = p_{02}, \alpha_{24} = p_{03}, -\alpha_{13} = p_{12}, -\alpha_{14} = p_{13}, \alpha_{13}\alpha_{24} - \alpha_{23}\alpha_{14} = p_{23}$. If we make the substitutions and homogenise the last equation with respect to p_{01}, we find the equation of the Klein quadric.

We remark that $G(r,n) \cap U_{i_1 \dots i_r}$ is the set of the subspaces W which are complementar to the subspace of equations $x_{i_1} = \dots = x_{i_r} = 0$.

Concluding, the projective algebraic set $G(r,n)$ has an affine open covering with irreducible varieties isomorphic to $\mathbb{A}^{r(n-r)}$, and it is easy to check that they have two by two non-empty intersection. Using Ex. 5 of §6, we deduce that $G(r,n)$ is a projective variety, of dimension $r(n-r)$, and it is rational.

In the special case $n \geq 4, r = 2$, using the Plücker coordinates $[\dots, p_{ij}, \dots]$, the equations of the Grassmannian $G(2,n)$ are of the form $p_{ij}p_{hk} - p_{ih}p_{jk} + p_{ik}p_{jh} = 0$, for any $i < j < h < k$.

Also in the case of $G(2,n)$, as for $\mathbb{P}^n \times \mathbb{P}^m$ and $V_{n,2}$, there is an interpretation in terms of matrices. Given a tensor in $\wedge^2 V$ with coordinates $[p_{ij}]$, we can consider the skew-symmetric $n \times n$ matrix whose term of position i,j is indeed p_{ij}, with the conditions $p_{ii} = 0$ and $p_{ji} = -p_{ij}$. In this way we can construct an isomorphism between $\wedge^2 V$ and the vector space of skew-symmetric matrices of order n.

From ${}^tA = -A$, it follows $\det(A) = (-1)^n \det(A)$. If n is odd, this implies $\det(A) = 0$. If n is even, one can prove that $\det(A)$ is a square. For instance if $n = 2$, and $A = \begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix}$, then $\det(A) = a^2$.

If $n = 4$, and $P = \begin{pmatrix} 0 & p_{12} & p_{13} & p_{14} \\ -p_{12} & 0 & p_{23} & p_{24} \\ -p_{13} & -p_{23} & 0 & p_{34} \\ -p_{14} & -p_{24} & -p_{34} & 0 \end{pmatrix}$, then $\det(P) = (p_{12}p_{34} - p_{13}p_{24} + p_{14}p_{23})^2$.

In general, for a skew-symmetric matrix A of even order $2n$, one defines the pfaffian of A, $pf(A)$, in one of the following equivalent ways:

(i) by recursion: if $n = 1$, $pf \begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix} = a^2$; if $n > 1$, one defines $pf(A) =$

$\Sigma_{i=2}^{2n}(-1)^i a_{1i} Pf(A_{1i})$, where A_{1i} is the matrix obtained from A removing the rows and the columns of indices 1 and i. Then one verifies that $pf(A)^2 = \det(A)$.

(ii) Given the matrix A, one considers the tensor $\omega = \Sigma_{i,j=1}^{2n} a_{ij} e_i \wedge e_j \in K^{2n}$. Then one defines the pfaffian as the unique constant such that $pf(A)e_1 \wedge \ldots \wedge e_{2n} = \frac{1}{n!}\omega \wedge \ldots \wedge \omega$.

For a skew-symmetric matrix of odd order, one defines the pfaffian to be 0.

10.13. Proposition. *A 2-tensor $\omega \in \wedge^2 V$ is totally decomposable if and only if $\omega \wedge \omega = 0$.*

Proof. If ω is decomposable, the conclusion easily follows. Conversely, if $\omega = \Sigma_{i,j=1}^{2n} a_{ij} e_i \wedge e_j$ and $\omega \wedge \omega = 0$, then the pfaffians of the principal minors of order 4 of the matrix A corresponding to ω are all 0, therefore from definition (ii) it follows that the pfaffians of the principal minors of all orders are 0, and also $\det(A) = 0$. In conclusion A has rank 2. Then one checks that ω is the $\wedge$ product of two vectors corresponding to two linearly independent rows of A. For instance, if $a_{12} \neq 0$, then $\omega = (a_{12}e_2 + \ldots + a_{1n}e_n) \wedge (-a_{12}e_1 + a_{23}e_3 + \ldots + a_{2n}e_n)$. □

The equations of $G(2,n)$ are the pfaffians of the principal minors of order 4 of the matrix P. They are all zero if and only if the rank of P is 2. Therefore the points of the Grassmannian $G(2,n)$, for any n, can be interpreted as skew-symmetric matrices of order n and rank 2.

The subvarieties of the Grassmannian $\mathbb{G}(r,n)$ correspond to subvarieties of $\mathbb{P}^n$ covered by linear spaces of dimension r. Conversely, any subvariety of $\mathbb{P}^n$ covered by linear spaces of dimension r gives rise to a subvariety of the Grassmannian.

10.14. Examples. 1. Pencils of lines. A pencil of lines in $\mathbb{P}^n$ is the set of lines passing through a fixed point O and contained in a 2-plane π such that $O \in \pi$. Assume that O has coordinates $[y_0, \ldots, y_n]$, and fix two points $A, B \in \pi$, different from O. Let $A = [a_0, \ldots, a_n]$, $B[b_0, \ldots, b_n]$. Then a general line of the pencil is generated by O and by a point of coordinates $[\ldots, \lambda a_i + \mu b_i, \ldots]$. Therefore the Plücker coordinates of a general line of the pencil are $p_{ij} = y_i(\lambda a_j + \mu b_j) - y_j(\lambda a_i + \mu b_i) = \lambda q_{ij} + \mu q'_{ij}$, where q_{ij}, q'_{ij} are the Plücker coordinates of the lines OA and OB respectively. So the lines of the pencil are represented in the Grassmannian by the points of a line. Conversely one can check that any line contained in a Grassmannian of lines represents the lines of a pencil.

2. Lines a smooth quadric surface. Let $\Sigma : x_0x_3 - x_1x_2 = \det\begin{pmatrix} x_0 & x_1 \\ x_2 & x_3 \end{pmatrix} = 0$ be the Segre quadric in $\mathbb{P}^3$. A line of the first ruling of Σ is characterised by a constant ratio of the rows of the matrix $\begin{pmatrix} x_0 & x_1 \\ x_2 & x_3 \end{pmatrix}$. Therefore it can be generated by two points with coordinates $[x_0, x_1, 0, 0]$, $[0, 0, x_0, x_1]$. The Plücker coordinates of such a line are $[0, x_0^2, x_0x_1, x_0x_1, x_1^2, 0]$. This describes a conic contained in

$\mathbb{G}(1,3)$. Similarly, the lines of the second ruling describe the points of another conic, indeed the coordinates are $[x_0^2, 0, x_0x_2, -x_0x_2, 0, x_2^2]$. These two conics are disjoint and contained in disjoint planes.

3. One can prove that $\mathbb{G}(1,3)$ contains two families of planes, and no linear space of dimension > 2. The planes of one family correspond to stars of lines in $\mathbb{P}^3$ (lines of $\mathbb{P}^3$ through a fixed point), while the planes of the second family correspond to the lines contained in the planes of $\mathbb{P}^3$. The geometry of the lines in $\mathbb{P}^3$ translates to give a decription of the geometry of the planes contained in $\mathbb{G}(1,3)$. Since on an algebraically closed field of characteristic $\neq 2$ two quadric hypersurfaces are projectively equivalent if and only if they have the same rank, one obtains a description of the geometry of all quadrics of maximal rank in $\mathbb{P}^5$.

Exercises to §10.

1. Using Ex. 5 of §6, prove that, if $X \subset \mathbb{P}^n$, $Y \subset \mathbb{P}^m$ are irreducible projective varieties, then $X \times Y$ is irreducible.

2. (*) Let $X \subset \mathbb{A}^n$, $Y \subset \mathbb{A}^n$. Show that $X \cap Y \simeq (X \times Y) \cap \Delta_{\mathbb{A}^n}$, where $\Delta_{\mathbb{A}^n}$ is the diagonal subvariety.

3. Let L, M, N be the following lines in $\mathbb{P}^3$:

$$L : x_0 = x_1 = 0, M : x_2 = x_3 = 0, N : x_0 - x_2 = x_1 - x_3 = 0.$$

Let X be the union of lines meeting L, M and N: write equations for X and describe it: is it a projective variety? If yes, of what dimension and degree?

4. Let X, Y be quasi–projective varieties, identify $X \times Y$ with its image via the Segre map. Check that the two projection maps $X \times Y \stackrel{p_1}{\to} X$, $X \times Y \stackrel{p_2}{\to} Y$ are regular. (Hint: use the open covering of the Segre variety by the Σ^{ij}'s.)

11. The dimension of an intersection.

Our aim in this section is to prove the following theorem:

11.1. Theorem. *Let K be an algebraically closed field. Let $X, Y \subset \mathbb{P}^n$ be quasi–projective varieties. Assume that $X \cap Y \neq \emptyset$. Then if Z is any irreducible component of $X \cap Y$, then* $\dim Z \geq \dim X + \dim Y - n$.

The proof uses in an essential way the Krull's principal ideal theorem (see for instance Atiyah–MacDonald [1]).

The proof of Theorem 11.1 will be divided in three steps. Note first that we can assume that $X \cap Y$ intersects $U_0 \simeq \mathbb{A}^n$, so, possibly after restricting X and Y, we may work with closed subsets of the affine space. Put $r = \dim X$, $s = \dim Y$.

Step 1. Assume that $X = V(F)$ is an irreducible hypersurface, with F irreducible polynomial of $K[x_1, \dots, x_n]$. The irreducible components of $X \cap Y$ correspond, by the Nullstellensatz, to the minimal prime ideals containing $I(X \cap Y)$

in $K[x_1,\dots,x_n]$. Let me recall that $I(X\cap Y)=\sqrt{I(X)+I(Y)}=\sqrt{\langle I(Y),F\rangle}$. So those prime ideals are the minimal ones over $\langle I(Y),F\rangle$. They correspond bijectively to minimal prime ideals containing $\langle f\rangle$ in $\mathcal{O}(Y)$, where f is the regular function on Y defined by F. We distinguish two cases:

- if $Y\subset X=V(F)$, then $f=0$ and $Y\cap X=Y$; $s=\dim Y>r+s-n=(n-1)+s-n$. So the theorem is true.

- if $Y\not\subset X$, then $f\neq 0$, moreover f is not invertible, otherwise $X\cap Y=\emptyset$: hence the minimal prime ideals over $\langle f\rangle$ in $\mathcal{O}(Y)$ have all height one, so for all Z, irreducible component of $X\cap Y$, $\dim Z=\dim Y-1=r+s-n$ (Theorem 7.7).

Step 2. Assume that $I(X)$ is generated by $n-r$ polynomials (where $n-r$ is the codimension of X): $I(X)=\langle F_1,\dots,F_{n-r}\rangle$. Then we can argue by induction on $n-r$: we first intersect Y with $V(F_1)$, whose irreducible components are all hypersurfaces, and apply Step 1: all irreducible components of $Y\cap V(F_1)$ have dimension either s or $s-1$. Then we intersect each of these components with $V(F_2)$, and so on. We conclude that every irreducible component Z has $\dim Z\geq \dim Y-(n-r)=r+s-n$.

Step 3. We use the isomorphism $\psi: X\cap Y\simeq (X\times Y)\cap\Delta_{\mathbb{A}^n}$ (see Ex.2, §10). Note that $X\times Y$ is irreducible by Proposition 6.11. ψ preserves the irreducible components and their dimensions, so we consider instead of X and Y, the algebraic sets $X\times Y$ and $\Delta_{\mathbb{A}^n}$, contained in $\mathbb{A}^{2n}$. We have $\dim X\times Y=r+s$ (Proposition 7.10). $\Delta_{\mathbb{A}^n}$ is a linear subspace of $\mathbb{A}^{2n}$, so it satisfies the assumption of Step 2; indeed it has dimension n in $\mathbb{A}^{2n}$ and is defined by n linear equations. Hence, for all Z we have: $\dim Z\geq (r+s)+n-2n=r+s-n$. □

The above theorem can be seen as a generalization of the Grassmann relation for linear subspaces. It is not an existence theorem, because it says nothing about $X\cap Y$ being non–empty. But for projective varieties, the following more precise version of the theorem holds:

11.2. Theorem. *Let $X,Y\subset\mathbb{P}^n$ be projective varieties of dimensions r,s. If $r+s-n\geq 0$, then $X\cap Y\neq\emptyset$.*

Proof. Let $C(X)$, $C(Y)$ be the affine cones associated to X and Y. Then $C(X)\cap C(Y)$ is certainly non–empty, because it contains the origin $O(0,0,\dots,0)$. Assume we know that $C(X)$ has dimension $r+1$ and $C(Y)$ has dimension $s+1$: then by Theorem 11.1 all irreducible components Z of $C(X)\cap C(Y)$ have dimension $\geq (r+1)+(s+1)-(n+1)=r+s-n+1\geq 1$, hence Z contains points different from O. These points give rise to points of $\mathbb{P}^n$ belonging to $X\cap Y$. It remains to show:

11.3. Proposition. *Let $Y\subset\mathbb{P}^n$ be a projective variety.*

Then $\dim Y=\dim C(Y)-1$. *If $S(Y)$ denotes the homogeneous coordinate ring, hence also* $\dim Y=\dim S(Y)-1$.

Proof. Let $p : \mathbb{A}^{n+1} \setminus \{O\} \to \mathbb{P}^n$ be the canonical morphism. Let us recall that $C(Y) = p^{-1}(Y) \cup \{O\}$. Assume that $Y_0 := Y \cap U_0 \neq \emptyset$ and consider also $C(Y_0) = p^{-1}(Y_0) \cup \{O\}$. Then we have:

$$C(Y_0) = \{(\lambda, \lambda a_1, \dots, \lambda a_n) \mid \lambda \in K, (a_1, \dots, a_n) \in Y_0\}.$$

So we can define a birational map between $C(Y_0)$ and $Y_0 \times \mathbb{A}^1$ as follows:

$$(y_0, y_1, \dots, y_n) \in C(Y_0) \to ((y_1/y_0, \dots, y_n/y_0), y_0) \in Y_0 \times \mathbb{A}^1,$$

$$((a_1, \dots, a_n), \lambda) \in Y_0 \times \mathbb{A}^1 \to (\lambda, \lambda a_1, \dots, \lambda a_n) \in C(Y_0).$$

Therefore $\dim C(Y_0) = \dim(Y_0 \times \mathbb{A}^1) = \dim Y_0 + 1$. To conclude, it is enough to remark that $\dim Y = \dim Y_0$ and $\dim C(Y) = \dim C(Y_0) = \dim S(Y)$. □

We observe that also $C(Y)$ and $Y \times \mathbb{P}^1$ are birationally equivalent.

11.4. Corollaries.
1. If $X, Y \subset \mathbb{P}^2$ are projective curves over an algebraically closed field, then $X \cap Y \neq \emptyset$.
2. $\mathbb{P}^1 \times \mathbb{P}^1$ is not isomorphic to $\mathbb{P}^2$.

Proof. 1. is a straightforward application of Theorem 11.2. To prove 2., assume by contradiction that $\phi : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^2$ is an isomorphism. If L, L' are skew lines on $\mathbb{P}^1 \times \mathbb{P}^1$, then $\phi(L)$, $\phi(L')$ are rational disjoint curves of $\mathbb{P}^2$, which contradicts 1.

If $X, Y \subset \mathbb{P}^n$ are varieties of dimensions r, s, then $r + s - n$ is called the *expected dimension* of $X \cap Y$. If all irreducible components Z of $X \cap Y$ have the expected dimension, then we say that the intersection $X \cap Y$ is *proper* or that *X and Y intersect properly.*

For example, two plane projective curves X, Y intersect properly if they don't have any common irreducible component. In this case, it is possible to predict the number of points of intersections. Precisely, it is possible to associate to every point $P \in X \cap Y$ a number $i(P)$, called the *multiplicity of intersection of X and Y at P*, in such a way that $\sum_{P \in X \cap Y} i(P) = dd'$, where d is the degree of X and d' is the degree of Y. This result is known as Theorem of Bézout, and is the first result of the branch of algebraic geometry called Intersection Theory. For a proof of the Theorem of Bézout, see for instance the classical book of Walker [8], or the book of Fulton on Algebraic Curves [5].

Let X be a closed subvariety of $\mathbb{P}^n$ (resp. of $\mathbb{A}^n$) of codimension r. X is called a *complete intersection* if $I_h(X)$ (resp. $I(X)$) is generated by r polynomials.

Hence, if X is a complete intersection of codimension r, then X is certainly the intersection of r hypersurfaces. Conversely, if X is intersection of r hypersurfaces, then, by Theorem 11.1, using induction, we deduce that $\dim X \geq n - r$; even assuming equality, we cannot conclude that X is a complete intersection, but simply that $I(X)$ is the radical of an ideal generated by r polynomials.

11.5. Example. Let $X \subset \mathbb{P}^3$ be the skew cubic. The homogeneous ideal of X is generated by the three polynomials F_1, F_2, F_3, the 2×2–minors of the matrix

$$M = \begin{pmatrix} x_0 & x_1 & x_2 \\ x_1 & x_2 & x_3 \end{pmatrix},$$

which are linearly independent polynomials of degree 2. Note that $I_h(X)$ does not contain any linear polynomial, because X is not contained in any hyperplane, and that the homogeneous component of minimal degree 2 of $I_h(X)$ is a vector space of dimension 3. Hence $I_h(X)$ cannot be generated by two polynomials, i.e. X is not a complete intersection.

Nevertheless, X is the intersection of the surfaces $V_P(F)$, $V_P(G)$, where

$$F = F_1 = \begin{vmatrix} x_0 & x_1 \\ x_1 & x_2 \end{vmatrix} \text{ and } G = \begin{vmatrix} x_0 & x_1 & x_2 \\ x_1 & x_2 & x_3 \\ x_2 & x_3 & x_0 \end{vmatrix}.$$

Clearly $F, G \in I_h(X)$ so $X \subset V_P(F) \cap V_P(G)$. Conversely, observe that $G = x_0F - x_3(x_0x_3 - x_1x_2) + x_2(x_1x_3 - x_2^2)$. If $P[x_0, \dots, x_3] \in V_P(F) \cap V_P(G)$, then P is a zero of $x_0x_3^2 - 2x_1x_2x_3 + x_2^3$, and therefore also of

$$x_2(x_0x_3^2 - 2x_1x_2x_3 + x_2^3) = x_1^2x_3^2 - 2x_1x_2^2x_3 + x_2^4 = (x_1x_3 - x_2^2)^2 = F_3^2.$$

Hence P is a zero also of $F_3 = x_1x_3 - x_2^2$. So P annihilates $x_3(x_0x_3 - x_1x_2) = x_3F_2$. If P satisfies the equation $x_3 = 0$, then it satisfies also $x_2 = 0$ and $x_1 = 0$, therefore $P = [1, 0, 0, 0] \in X$. If $x_3 \neq 0$, then $P \in V_P(F_1, F_2, F_3) = X$.

The geometric description of this phenomenon is that the skew cubic X is the set-theoretic intersection of a quadric and a cubic, which are tangent along X, so their intersection is X counted with multiplicity 2.

This example motivates the following definition: X is a *set–theoretic complete intersection* if $\operatorname{codim} X = r$ and the ideal of X is the radical of an ideal generated by r polynomials. It is an open problem if all irreducible curves of $\mathbb{P}^3$ are set–theoretic complete intersections. For more details, see [4].

Exercises to §11.

1. Let $X \subset \mathbb{P}^2$ be the union of three points not lying on a line. Prove that the homogeneous ideal of X cannot be generated by two polynomials.

12. Complete varieties.

We work over an algebraically closed field K.

12.1. Definition. Let X be a quasi–projective variety. X is *complete* if, for any quasi–projective variety Y, the natural projection on the second factor $p_2 : X \times Y \to Y$ is a closed map. (Note that both projections p_1, p_2 are morphisms: see Exercise 4 to §10.)

Example. The affine line $\mathbb{A}^1$ is not complete: let $X = Y = \mathbb{A}^1$, $p_2 : \mathbb{A}^1 \times \mathbb{A}^1 = \mathbb{A}^2 \to \mathbb{A}^1$ is the map such that $(x_1, x_2) \to x_2$. Then $Z := V(x_1x_2 - 1)$ is closed in $\mathbb{A}^2$ but $p_2(Z) = \mathbb{A}^1 \setminus \{O\}$ is not closed.

12.2. Proposition. *(i) If $f : X \to Y$ is a regular map and X is complete, then $f(X)$ is a closed complete subvariety of Y.*

(ii) If X is complete, then all closed subvarieties of X are complete.

Proof. (i) Let $\Gamma_f \subset X \times Y$ be the graph of f: $\Gamma_f = \{(x, f(x)) \mid x \in X\}$. It is clear that $f(X) = p_2(\Gamma_f)$, so to prove that $f(X)$ is closed it is enough to check that Γ_f is closed in $X \times Y$. Let us consider the diagonal of Y: $\Delta_Y = \{(y, y) \mid y \in Y\} \subset Y \times Y$. If $Y \subset \mathbb{P}^n$, then $\Delta_Y = \Delta_{\mathbb{P}^n} \cap (Y \times Y)$, so it is closed because $\Delta_{\mathbb{P}^n}$ is the closed subset defined in $\Sigma_{n,n}$ by the equations $w_{ij} - w_{ji} = 0$, $i, j = 0, \ldots, n$. There is a natural map $f \times 1_Y : X \times Y \to Y \times Y$, $(x, y) \to (f(x), y)$, such that $(f \times 1_Y)^{-1}(\Delta_Y) = \Gamma_f$. It is easy to see that $f \times 1_Y$ is regular, so Γ_f is closed, so also $f(X)$ is closed.

Let now Z be any variety and consider $p_2 : f(X) \times Z \to Z$ and the regular map $f \times 1_Z : X \times Z \to f(X) \times Z$. There is a commutative diagram:

$$\begin{array}{ccc} X \times Z & \xrightarrow{p_2'} & Z \\ \downarrow {\scriptstyle f \times 1_Z} & \nearrow & {\scriptstyle p_2} \\ f(X) \times Z & & \end{array}$$

If $T \subset f(X) \times Z$, then $(f \times 1_Z)^{-1}(T)$ is closed and $p_2(T) = p_2'((f \times 1_Z)^{-1}(T))$ is closed because X is complete. We conclude that $f(X)$ is complete.

(ii) Let $T \subset X$ be a closed subvariety and Y be any variety. We have to prove that $p_2 : T \times Y \to Y$ is closed. If $Z \subset T \times Y$ is closed, then Z is closed also in $X \times Y$, hence $p_2(Z)$ is closed because X is complete. □

12.3. Corollaries.

1. If X is a complete variety, then $\mathcal{O}(X) \simeq K$.

2. If X is an affine complete variety, then X is a point.

Proof. 1. If $f \in \mathcal{O}(X)$, f can be interpreted as a regular map $f : X \to \mathbb{A}^1$. By Proposition 12.2, (i), $f(X)$ is a closed complete subvariety of $\mathbb{A}^1$, which is not complete. Hence $f(X)$ has dimension < 1 and is irreducible, hence it is a point, so $f \in K$.

2. By 1., $\mathcal{O}(X) \simeq K$. But $\mathcal{O}(X) \simeq K[x_1, \ldots, x_n]/I(X)$, hence $I(X)$ is maximal. By the Nullstellensatz, X is a point. □

12.4. Theorem. *Let X be a projective variety. Then X is complete.*

Proof. (sketch, see Šafarevič [7].)

1. It is enough to prove that $p_2 : \mathbb{P}^n \times \mathbb{A}^m \to \mathbb{A}^m$ is closed, for all n, m. This can be observed by using the local character of closedness and the affine open coverings of quasi–projective varieties.

2. If $x_0, \ldots, x_n$ are homogeneous coordinates on $\mathbb{P}^n$ and $y_1, \ldots, y_m$ are coordinates on $\mathbb{A}^m$, then any closed subvariety of $\mathbb{P}^n \times \mathbb{A}^m$ can be characterised as the set of common zeroes of a set of polynomials in the variables $x_0, \ldots, x_n, y_1, \ldots, y_m$, homogeneous in the first group of variables $x_0, \ldots, x_n$.

3. Let $Z \subset \mathbb{P}^n \times \mathbb{A}^m$ be closed. Then Z is the set of solutions of a system of equations

$$\{G_i(x_0, \ldots, x_n; y_1, \ldots, y_m) = 0, i = 1, \ldots, t$$

where G_i is homogeneous in the x's. A point $P(\overline{y}_1, \ldots, \overline{y}_m)$ is in $p_2(Z)$ if and only if the system

$$\{G_i(x_0, \ldots, x_n; \overline{y}_0, \ldots, \overline{y}_m) = 0, i = 1, \ldots, t$$

has a solution in $\mathbb{P}^n$, i.e. if the ideal of $K[x_0, \ldots, x_n]$ generated by $G_1(x; \overline{y}), \ldots, G_t(x; \overline{y})$ has at least one zero in $\mathbb{P}^n$. Hence

$$\begin{aligned} p_2(Z) &= \{(\overline{y}_1, \ldots, \overline{y}_m) | \ \forall\ d \geq 1\ \langle G_1(x; \overline{y}), \ldots, G_t(x; \overline{y})\rangle \not\supset K[x_0, \ldots, x_n]_d\} = \\ &= \bigcap_{d \geq 1} \{(\overline{y}_1, \ldots, \overline{y}_m) | \ \langle G_1(x; \overline{y}), \ldots, G_t(x; \overline{y})\rangle \not\supset K[x_0, \ldots, x_n]_d\}. \end{aligned}$$

Let $\{M_\alpha\}_{\alpha=1,\ldots,\binom{n+d}{d}}$ be the set of the monomials of degree d in $K[x_0, \ldots, x_n]$; let $d_i = \deg G_i(x; \overline{y})$; let $\{N_i^\beta\}$ be the set of the monomials of degree $d - d_i$; let finally $T_d = \{(\overline{y}_1, \ldots, \overline{y}_m) | \ \langle G_1(x; \overline{y}), \ldots, G_t(x; \overline{y})\rangle \not\supset K[x_0, \ldots, x_n]_d\}$.

Then $P(\overline{y}_1, \ldots, \overline{y}_m) \notin T_d$ if and only if $M_\alpha = \sum_i G_i(x; \overline{y}) F_{i,\alpha}(x_0, \ldots, x_n)$, for all α and for suitable polynomials $F_{i,\alpha}$ homogeneous of degree $d - d_i$. So $P \notin T_d$ if and only if, for all index α, M_α is a linear combination of the polynomials $\{G_i(x; \overline{y}) N_i^\beta\}$, i.e. the matrix A of the coefficients of the polynomials $G_i(x; \overline{y}) N_i^\beta$ with respect to the basis $\{M_\alpha\}$ has maximal rank $\binom{n+d}{d}$. So T_d is the set of zeroes of the minors of a fixed order of the matrix A, hence it is closed. □

12.5. Corollary. *Let X be a projective variety. Then $\mathcal{O}(X) \simeq K$.*

12.6. Corollary. *Let X be a projective variety, $\phi : X \to Y \subset \mathbb{P}^n$ be any regular map. Then $\phi(X)$ is a projective variety. In particular, if $X \simeq Y$, then Y is projective.*

13. The tangent space.

We define the tangent space $T_{X,p}$ at a point P of an *affine* variety X as the union of the lines passing through P and touching X at P. Then we will find a "local" characterization of $T_{X,p}$, only depending on the local ring $\mathcal{O}_{X,p}$: this will allow to define the tangent space at a point of any quasi–projective variety.

Assume first that $X \subset \mathbb{A}^n$ is closed and $P = (0, \ldots, 0)$. Let L be a line through P: if $A(a_1, \ldots, a_n)$ is another point of L, then a general point of L has