Exercises QFT II — 2018/2019

Problem Sheet 8

Problem 15: Grassmann numbers

Consider the Grassmann variables $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ and $\beta = (\beta_1, \beta_2, \beta_3)$ and the Grassmann integration measure

$$d\alpha d\beta \equiv d\alpha_1 d\beta_1 d\alpha_2 d\beta_2 d\alpha_3 d\beta_3 .$$

Remember the integration rules:

$$\int d\alpha_j = \int d\beta_k = 0 \qquad \forall j,k \qquad\qquad \int d\alpha_j\, \alpha_j = \int d\beta_k\, \beta_k = 1 \qquad \forall j,k$$

Consider a 3×3 matrix

$$M = \begin{pmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{pmatrix} \tag{1}$$

1. Compute $\det M$.
2. Show that
$$\int d\alpha\, d\beta\, e^{\alpha^T \cdot M \cdot \beta} = \det M\ . \tag{2}$$
3. Compute the element $(M^{-1})^{31}$ of the matrix M^{-1} (inverse matrix of M).
4. Show that
$$\int d\alpha\, d\beta\, \alpha_1\, \beta_3\, e^{\alpha^T \cdot M \cdot \beta} = \sigma (M^{-1})^{31} \det M\ , \tag{3}$$
where σ is a constant to determine.

Problem 16 - Propagator in the axial gauge.

During the lectures, we have seen that the gauge can be fixed consistently in the path integral, simply by adding to the Lagrangian

$$\mathcal{L}_{\text{g.f.}} = -\frac{1}{2\zeta}\left(\partial^\mu A_\mu\right)^2 \ . \tag{4}$$

The free field genereting functional can then be written as

$$Z[j] = \int \mathcal{D}A e^{i\int d^4x \left\{\frac{1}{2}A_\mu \mathbf{K}^{\mu\nu}(x) A_\nu + j_\mu A^\mu\right\}} \qquad \text{with} \qquad \mathbf{K}^{\mu\nu}(x) \equiv \eta^{\mu\nu}\partial^2 - \left(1 - \frac{1}{\zeta}\right)\partial^\mu\partial^\nu \tag{5}$$

The Propagator is then the inverse of the operator $\mathbf{K}^{\mu\nu}$ and it is given in momentum space by

$$\Delta_F^{\mu\nu}(k) = \frac{-1}{k^2 + i\epsilon}\left[\eta^{\mu\nu} - (1-\zeta)\frac{k^\mu k^\nu}{k^2}\right] \ . \tag{6}$$

We want to repeat now the computation of the propagator, but in the axial gauge, that it is defined by the gauge fixing Lagrangian

$$\mathcal{L}_{\text{g.f.}} = \frac{1}{2\xi}\left(n^\mu A_\mu\right)^2 \ . \tag{7}$$

1. What is the gauge fixing condition $G(A) = 0$ that produces the Lagrangian (7)? Why can we neglect the Fadeev-Popov determinant also for the axial case?

2. Find the expression of $\mathbf{K}^{\mu\nu}$ in the axial gauge.

3. Show that the gauge boson propagator in momentum space is now given by

$$\Delta_F^{\mu\nu}(k) = \frac{-1}{k^2 + i\epsilon}\left[\eta^{\mu\nu} - \frac{n^\mu k^\nu + k^\mu n^\nu}{k \cdot n} - \frac{(\xi\, k^2 - n^2)\, k^\mu k^\nu}{(n \cdot k)^2}\right] \tag{8}$$

4. The difference with respect to the propagator in the covariant gauge (4) is proportional to k^μ and/or k^ν, $\forall \zeta, \xi$. Is this a surprise? Why?