Exercises QFT II — 2018/2019

Problem Sheet 9

Problem 17 - Ward identities in QED.

The generating functional for QED in the covariant gauge is

$$Z[j, \eta.\bar{\eta}] = \int \mathcal{D}A\mathcal{D}\psi\mathcal{D}\bar{\psi} e^{i\int d^4x\,\mathcal{L}_{\text{g.f.}} + i\int d^4x\, j^\mu A_\mu + i\int d^4x\,(\bar{\psi}\eta + \bar{\eta}\psi)}\,, \tag{1}$$

with

$$\mathcal{L}_{\text{g.f.}} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \bar{\psi}\left(i\gamma^\mu(\partial_\mu - ieA_\mu) - m\right)\psi - \frac{1}{2\zeta}\left(\partial^\mu A_\mu\right)^2 \tag{2}$$

1. Show that if the measure $\mathcal{D}A\mathcal{D}\psi\mathcal{D}\bar{\psi}$ in (1) is invariant under the (infinitesimal) gauge transformations

   $$A_\mu \mapsto A_\mu + \partial_\mu\alpha \qquad \text{and} \qquad \psi \mapsto \psi - ie\alpha\psi\,, \tag{3}$$

   we obtain the **Ward-Takahashi identity**

   $$\left[\frac{i}{\zeta}\Box\partial^\mu\frac{\delta}{\delta j_\mu(x)} - \partial^\mu j_\mu(x) - e\left(\bar{\eta}\frac{\delta}{\delta\bar{\eta}} - \eta\frac{\delta}{\delta\eta}\right)\right] Z[j, \eta\bar{\eta}] = 0\,. \tag{4}$$

   [Use the invariance of the integral under the change of variables (3).]

2. Translate the identity (4) into an equation for the generating functional for the connected Green's functions W. [Remember that $Z = e^{iW}$.]

3. Consider the effective action

   $$\Gamma[\Psi, \bar{\Psi}, \mathcal{A}] = W[\eta, \bar{\eta}, j] - \int d^4x\,(j^\mu\mathcal{A}_\mu + \bar{\Psi}\eta + \bar{\eta}\Psi)\,, \tag{5}$$

   where

   $$\Psi = \frac{\delta W}{\delta\bar{\eta}}\,, \qquad \bar{\Psi} = \frac{\delta W}{\delta\eta}\,, \qquad \mathcal{A}_\mu = \frac{\delta W}{\delta j^\mu}\,. \tag{6}$$

   Show that the identity (4) can be written as the following equation for Γ:

   $$-\frac{1}{\zeta}\Box(\partial^\mu\mathcal{A}_\mu) + \partial_\mu\frac{\delta\Gamma}{\delta\mathcal{A}_\mu} - ie\left(-\Psi\frac{\delta\Gamma}{\delta\Psi} + \bar{\Psi}\frac{\delta\Gamma}{\delta\bar{\Psi}}\right) = 0\,. \tag{7}$$

   [Remember that $\eta = -\frac{\delta\Gamma}{\delta\bar{\Psi}}$, $\bar{\eta} = -\frac{\delta\Gamma}{\delta\Psi}$ and $j^\mu = -\frac{\delta\Gamma}{\delta\mathcal{A}_\mu}$.]

4. Show that by differentiating (7) with respect to $\bar{\Psi}(x)$ and $\Psi(y)$ and setting $\Psi = \bar{\Psi} = \mathcal{A}_\mu = 0$ one can derive the familiar Ward Identity

   $$k^\mu\Gamma_\mu(p, k, p+k) = S_F^{-1}(p+k) - S_F^{-1}(p)\,, \tag{8}$$

where Γ_μ and S_F^{-1} are obtained from:

$$\int d^4z d^4x d^4y\, e^{i(p'x-py-kx)} \frac{\delta^3\Gamma}{\delta\bar{\Psi}(x)\delta\Psi(y)\delta\mathcal{D}^\mu(z)} = ie(2\pi)^4\delta^4(p'-p-k)\Gamma_\mu(p,k,p') \tag{9}$$

and

$$\int d^4x d^4y\, e^{i(p'x-py)} \frac{\delta^2\Gamma}{\delta\bar{\Psi}(x)\delta\Psi(y)} = i(2\pi)^4\delta^4(p'-p)iS_F^{-1}(p)\,. \tag{10}$$

5. From the answer to question 2, show that

$$\frac{1}{\zeta}\Box_x\partial_x^\mu \frac{\delta^2 W}{\delta j_\mu(x)\delta j_\nu(y)} = \partial_x^\mu g_{\mu\nu}\delta^4(x-y) \tag{11}$$

   (where $g_{\mu\nu}$ is the Minkowski metric). What is the corresponding relation in momentum space?

Problem 18: Scalar QED

Consider a complex scalar theory, described by the Lagrangian:

$$\mathcal{L} = \partial_\mu\Phi^*\partial^\mu\Phi - m^2\Phi^*\Phi\,. \tag{12}$$

1. By using the functional method, show that the propagator for the complex scalar is still

$$i\Delta_F(p) = \frac{i}{p^2-m^2+i\epsilon}\,. \tag{13}$$

2. Show that the Lagrangian (12) is invariant under the global symmetry transformation $\Phi \mapsto e^{i\alpha}\Phi$.

3. Find the expression of the covariant derivative D_μ, such that the Lagrangian

$$(D_\mu\Phi)^*(D^\mu\Phi) - m^2\Phi^*\Phi \tag{14}$$

   is invariant under the gauge transformation $\Phi \mapsto e^{i\,e\,\alpha(x)}\Phi$.

Add the kinetic term for the gauge boson A_μ to obtain the Lagrangian for the Scalar QED

$$\mathcal{L}_{\rm sQED} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + (D_\mu\Phi)^*(D^\mu\Phi) - m^2\Phi^*\Phi \qquad \text{with } F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu\,. \tag{15}$$

4. The interaction contains a term that takes the form $A_\mu j_s^\mu$. Write the expression for j_s^μ. Show that j_s^μ transforms as a vector under Lorentz transformations.

5. Derive the Feynman rules for the Scalar QED. How many interaction vertices are there?

6. Draw the Feynman diagrams that represent the quantum corrections to the scalar propagator at order e^2.