Exercises QFT II — 2018/2019

Problem Sheet 11

Problem 21 - Non-abelian gauge theories

Solve the following problems:

1. Consider the Lie group $SU(N)$ of $N \times N$ unitary matrices with det $= 1$. Show that you can parametrize this space by $N^2 - 1$ parameters. To do this, derive the corresponding Lie Algebra and find out the generators. Show that the maximum number of mutually commuting generators is $N - 1$.

2. Given a Lie algebra $\mathcal{G}$ with $D = \dim \mathcal{G}$defined by
$$[T^a, T^b] = i f^{abc} T^c \qquad\qquad a, b, c = 1, ..., D\,. \tag{1}$$
The f^{abc} can be regarded as the coefficients of D $D \times D$ matrices $\mathbf{f}^a$ (i.e. $(\mathbf{f}^a)^{bc} \equiv f^{abc}$). Using the Jacobi identity, show that these matrices $\mathbf{f}^a$ obey the same commutation relations as the abstract generators T^a (when a proper factor of -1 is provided).

3. Consider the Lagrangian
$$\mathcal{L} = \partial_\mu \Phi^\dagger \cdot \partial^\mu \Phi - m \Phi^\dagger \cdot \Phi \tag{2}$$
where Φ is a field in the fundamental representation of $SU(N)$. Show that $\mathcal{L}$ is invariant under the $SU(N)$ transformations $\Phi \mapsto U \cdot \Phi$ (with $U \in SU(N)$ a constant matrix). Rewrite everything in components.

4. Consider the field φ in the Adjoint representation of $SU(N)$. Show that its covariant derivative is given by
$$\mathbf{D}_\mu \varphi = \partial_\mu \varphi + [\mathbb{A}, \varphi] \tag{3}$$
i.e. show that $\mathbf{D}_\mu \varphi$ transforms in the same way as φ.

Problem 22 - Fermion two-point function in non-abelian gauge theories

Consider a gauge theory with non-abelian group G and a fermion ψ in the representation R of this group[1]:

$$\mathcal{L} = -\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} + \bar{\psi}\left[i\gamma^\mu(\partial_\mu + iA^a_\mu t^a_R) - m\right]\psi \tag{4}$$

1. Write down the Feynman rules that are necessary for computing the graph in Figure 1 (take the gauge boson propagator in the Feynman gauge).
2. Write down the integral corresponding to the graph in Figure 1.
3. Compute the divergent part.

[Remember that $\sum_a t^a_R t^a_R = C_2(R)\,\mathbf{1}_{\dim R}$.]

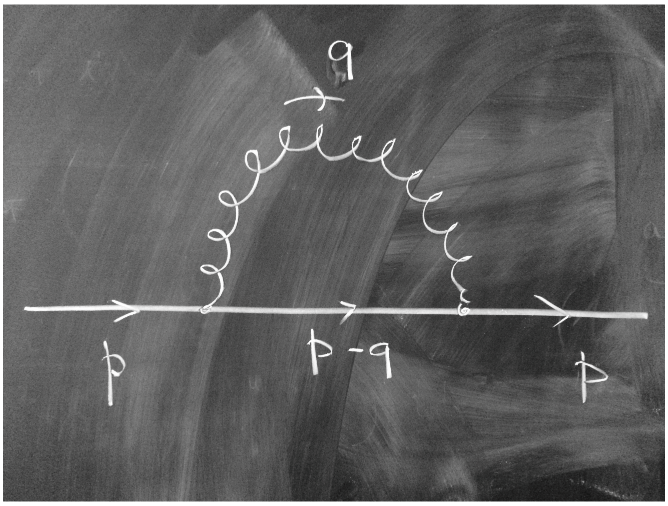

Figure 1: One-loop fermionic two-point function.

[1] Remember that this means that ψ is a vector belonging to the vector space where the matrices t^a_R act. Each component of this vector is a Dirac spinor.