Exercises QFT II — 2018/2019

Problem Sheet 12

Problem 23 - Anomaly and Ward Identities I

Consider the free Lagrangian

$$\mathcal{L} = \bar{\psi}(i\,\not{\partial} - m)\psi\,, \tag{1}$$

and the following currents:

$$j^\mu = \bar{\psi}\gamma^\mu\psi \qquad\qquad j_5^\mu = \bar{\psi}\gamma^\mu\gamma_5\psi \tag{2}$$

1. Using the equation of motion, compute the classical value for $\partial_\mu j^\mu$ and $\partial_\mu j_5^\mu$. What happens when $m = 0$?

Consider the following (Fourier transform of the) Green function:

$$T_{\mu\nu\lambda} = i\int d^4x\, d^4y\, d^4z\, e^{ik_1x+ik_2y-iqz}\ \langle 0|Tj_\mu(x)j_\nu(y)j_\lambda^5(z)|0\rangle \tag{3}$$

2. Draw the (triangle) Feynmann diagrams that describe this Green function.
3. Using Feynman rules, show that

$$T_{\mu\nu\lambda}(m) = i\int \frac{d^4p}{(2\pi)^4}(-)\mathrm{tr}\frac{i}{\not{p}-m}\gamma_\lambda\gamma_5\frac{i}{\not{p}-\not{q}-m}\gamma_\nu\frac{i}{\not{p}-\not{k}_1-m}\gamma_\mu + \begin{pmatrix} k_1 \leftrightarrow k_2 \\ \mu \leftrightarrow \nu \end{pmatrix} \tag{4}$$

Consider also

$$T_{\mu\nu} = i\int d^4x\, d^4y\, d^4z\, e^{ik_1x+ik_2y-iqz}\langle 0|Tj_\mu(x)j_\nu(y)P(z)|0\rangle \qquad \text{with } P = \bar{\psi}\gamma_5\psi \tag{5}$$

4. Show that plugging the classical results for $\partial_\mu j^\mu$ and $\partial_\mu j_5^\mu$ into (3), one obtains:

   $$\text{AWI}:\ q^\lambda T_{\mu\nu\lambda} = 2mT_{\mu\nu} \qquad\qquad \text{VWI}:\ k_1^\mu T_{\mu\nu\lambda} = k_2^\nu T_{\mu\nu\lambda} = 0 \tag{6}$$

   These are called the Axial Ward Identity and the Vector Ward Identity.

5. Draw the Feynman diagrams for $T_{\mu\nu}$ and show that:

$$T_{\mu\nu}(m) = i\int \frac{d^4p}{(2\pi)^4}(-)\mathrm{tr}\frac{i}{\not{p}-m}\gamma_5\frac{i}{\not{p}-\not{q}-m}\gamma_\nu\frac{i}{\not{p}-\not{k}_1-m}\gamma_\mu + \begin{pmatrix} k_1 \leftrightarrow k_2 \\ \mu \leftrightarrow \nu \end{pmatrix} \tag{7}$$

6. Prove that this expression is convergent. [Write down explicitly the fermion propagator and use the properties of the traces of the gamma matrices with γ_5.]

Let us consider the Axial Ward Identity and see if the classical result is the same as the quantum one.

7. Using $\not{q}\gamma_5 = \gamma_5(\not{p} - \not{q} - m) + (\not{p} - m)\gamma_5 + 2m\gamma_5$, show that

$$q^\lambda T_{\mu\nu\lambda} = 2mT_{\mu\nu} + R^1_{\mu\nu} + R^2_{\mu\nu} \tag{8}$$

where

$$R^1_{\mu\nu} = \int \frac{d^4p}{(2\pi)^4}[f(p-k_2) - f(p)] \qquad \text{with} \qquad f(p) = \mathrm{tr}\,\frac{1}{\not{p}-m}\gamma_5\gamma_\nu\frac{1}{\not{p}-\not{k}_1-m}\gamma_\mu \tag{9}$$

and $R^2_{\mu\nu}$ is equal to $R^1_{\mu\nu}$ with $k_1 \leftrightarrow k_2$ and $\mu \leftrightarrow \nu$.

8. Prove that $R^1_{\mu\nu}$ is linearly divergent. Then we need to regularize it.

Problem 24 - Anomaly and Ward Identities II

Pauli-Villars regularization: We introduce a regulator that preserves the Vector WI, while it violates the Axial WI (there exsists no regulator that preserves both). We define:

$$T^{\mathrm{reg}}_{\mu\nu\lambda} \equiv T_{\mu\nu\lambda}(m) - T_{\mu\nu\lambda}(m = M) \qquad \text{and} \tag{10}$$

we take the difference before doing the integral. Then the physical Green function is defined as: $T^{\mathrm{phys}}_{\mu\nu\lambda} \equiv \lim_{M\to\infty} T^{\mathrm{reg}}_{\mu\nu\lambda}$

1. Prove that $T^{\mathrm{reg}}_{\mu\nu\lambda}$ is finite.
2. Prove that now $R^{1\mathrm{reg}}_{\mu\nu}$ is finite and actually zero.
   [Hint: Show first that $\int_{-\infty}^{+\infty} dx\,[f(x+a) - f(x)] = 0$ if $\int_{-\infty}^{+\infty} dx\, f(x)$ is convergent.]
3. Prove that the Vector WI is satisfied, i.e. $k_1^\mu T^{\mathrm{reg}}_{\mu\nu\lambda} = 0 = k_2^\nu T^{\mathrm{reg}}_{\mu\nu\lambda}$.
4. Show that the Axial WI is now:

$$q^\lambda T^{\mathrm{phys}}_{\mu\nu\lambda} = 2mT_{\mu\nu}(m) - \lim_{M\to\infty} 2MT_{\mu\nu}(M) \tag{11}$$

5. Show that $T_{\mu\nu}(M)$ can be written as

$$T_{\mu\nu}(M) = \int \frac{d^4p}{(2\pi)^4} 2\int_0^1 dx \int_0^{1-x} dy \frac{4iM\,\epsilon_{\mu\nu\alpha\beta}k_1^\alpha k_2^\beta}{[p^2 - 2p\cdot\tilde{k}_1 - \tilde{M}^2]^3} + \begin{pmatrix} 1 \leftrightarrow 2 \\ \mu \leftrightarrow \nu \end{pmatrix} \tag{12}$$

   with appropriate $\tilde{k}_1$ and $\tilde{M}$. Determine $\tilde{k}_1$ and $\tilde{M}_1$ explicitly.
6. Use the relation

$$\int \frac{d^np}{(p^2 - 2p\cdot\tilde{k} - \tilde{M}^2)^A} = i^{1-2A}\pi^{n/2}\frac{\Gamma(A-\frac{n}{2})}{\Gamma(A)}\frac{1}{(\tilde{k}^2+\tilde{M}^2)^{A-n/2}} \tag{13}$$

   to compute the integral (12).
7. Perform the limit $M \to \infty$ to show that

$$q^\lambda T^{\mathrm{phys}}_{\mu\nu\lambda} \;=\; 2mT_{\mu\nu}(m) - \tfrac{1}{2\pi^2}\epsilon_{\mu\nu\alpha\beta}k_1^\alpha k_2^\beta\,. \tag{14}$$
