$f = u + iv \in H(B_1(0))$, supp. $u \in C(\overline{B_1(0)})$ e

$v(0) = 0$. Provare

$$f(z) = \frac{1}{2\pi} \int_{\partial B_1(0)} \frac{\xi + z}{\xi - z} u(\xi)\,|d\xi| \quad , \forall z \in B_1(0).$$

Dim.

$$u(z) = \int_{\partial B_1(0)} P(z;\xi)\, u(\xi)\,|d\xi|$$

$$P(z;\xi) = \frac{\xi}{2\pi}\left(\frac{1}{\xi - z} - \frac{1}{\xi - \frac{1}{\bar z}}\right)$$

Fissato $\xi \in \partial B_1(0)$, abbiamo

$$P(z;\xi) = F(z) + \overline{G(z)}$$

dove

$$F(z) = \frac{\xi}{2\pi}\left(\frac{1}{\xi - z}\right)$$

$$G(z) = -\frac{\bar\xi}{2\pi}\left(\frac{1}{\bar\xi - \frac{1}{z}}\right)$$

P è a valori reali $\Rightarrow P = \overline{P}$

$$P(z;\zeta) = F + \overline{G} = \overline{F} + G =$$

$$= \frac{F+G}{2} + \frac{\overline{F+G}}{2} = \mathrm{Re}\,(F+G)$$

$$F(z) = \frac{\zeta}{2\pi}\,\frac{1}{\zeta - z}$$

$$G(z) = -\frac{\overline{\zeta}}{2\pi}\left(\frac{1}{\overline{\zeta} - \frac{1}{z}}\right) = -\frac{\overline{\zeta}}{2\pi}\,\frac{z}{\overline{\zeta} z - 1}$$

$$F + G = \frac{1}{2\pi}\left(\frac{\zeta}{\zeta - z} - \frac{\overline{\zeta} z}{\overline{\zeta} z - 1}\right) =$$

$$= \frac{1}{2\pi}\left(\frac{\zeta(\overline{\zeta} z - 1) - (\zeta - z)\overline{\zeta} z}{(\zeta - z)(\overline{\zeta} z - 1)}\right) =$$

$$= \frac{1}{2\pi}\,\frac{\cancel{z} - \zeta - \cancel{z} + \overline{\zeta} z^2}{} =$$

$$\shortparallel$$

$$= \frac{1}{2\pi} \frac{\bar{\xi} z^2 - \xi}{(\xi - z)(\bar{\xi} z - 1)} =$$

$$= \frac{1}{2\pi} \frac{\bar{\xi} z^2 - \xi}{z - \xi - \bar{\xi} z^2 + z} =$$

$$= \frac{1}{2\pi} \frac{z^2 - \xi^2}{2\xi z - \xi^2 - z^2} =$$

$$= \frac{1}{2\pi} \frac{(z+\xi)(z-\xi)}{-(z-\xi)^2} = \frac{1}{2\pi} \frac{\xi + z}{\xi - z}$$

$$u(z) = \int_{\partial B_1(0)} \operatorname{Re}\left(\frac{1}{2\pi} \frac{\xi + z}{\xi - z} \right) u(\xi) \, |d\xi| =$$

$$= \operatorname{Re}\left(\int_{\partial B_1(0)} \frac{1}{2\pi} \frac{\xi + z}{\xi - z} u(\xi) \, |d\xi| \right)$$

$$u = \operatorname{Re} f$$

$$\boxed{f(z) = \int_{\partial B_1(0)} \frac{1}{2\pi} \frac{\xi + z}{\xi - z} u(\xi) \, |d\xi|} + c$$

$v(0) = 0 \quad ! \quad f(0) = u(0)$

$$f(0) = \underbrace{\frac{1}{2\pi} \int_{\partial B_1(0)} u(\xi)\,|d\xi|}_{u(0)} + c \implies c = 0.$$

$\square$

Sia $f \in H(\mathbb{C})$ t.c.

$$|f(z)| \leq \frac{(1+|z|^5)\,|z^4-3|}{2+|z|^6}, \quad \forall z \in \mathbb{C}.$$

Provare che $f \equiv 0$.

Per $|z| \to \infty$

$$|f(z)| = O(|z|^3)$$

f è un polinomio di grado ≤ 3.

Ora $z^4 - 3$ ha 4 radici distinte

Quindi f si annulla in 4 punti distinti

$\implies f \equiv 0$ $\square$

$$f(z) = z^6 + z^5 - 64$$

Dire quanti zeri (contati con le molt.) ha f nell'insieme:

$$K = \{ z \in \mathbb{C} \mid 1 < |z| < 3 \} =$$

$$= B_3(0) \setminus \overline{B_1(0)}$$

$$g(z) = z^6 - 64$$

g ha 6 zeri tutti su $\partial B_2(0) \subset K$.

$$\partial K = \partial B_3(0) \cup \partial B_1(0)$$

$$f(z) - g(z) = z^5$$

$$|f-g|\big|_{\partial B_3(0)} = \underline{\underline{3^5}} = 81 \times 3 = 243.$$

$$|f-g|_{\partial B_1(0)} = 1$$

$g(z) = z^6 - 64$

su $\partial B_3(0)$

$$|g(z)| \geq 3^6 - 64 = 3(242) - 64 =$$

$$= 2(243) - |\cdots > 243$$

su $\partial B_1(0)$

$$|g(z)| = |z^6 - \underline{\underline{64}}| \geq 63 > 1 = |f - g|.$$

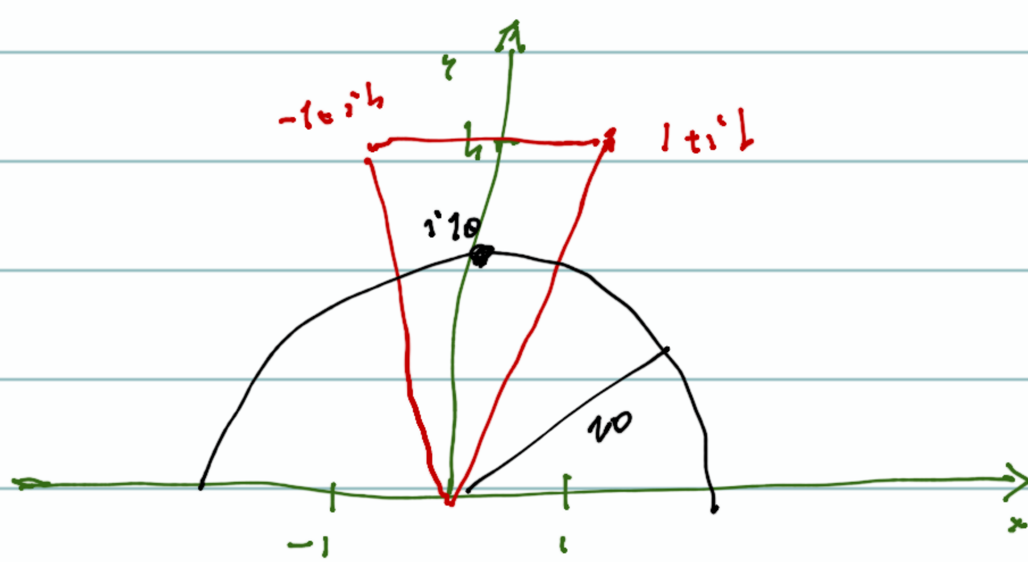
Quindi su tutto ∂K

$$|f - g| < |g|$$

f ha 6 zeri in K.

$h > 0$

$$\gamma_h = [0, 1 + ih] + [1 + ih, -1 + ih] + [-1 + ih, 0]$$



$$\lim_{h \to +\infty} \int_{\gamma_h} \underbrace{\frac{1}{z^8 + 10^8}} dz$$

$$(i\,10)^8 = i^8\, 10^8 = (i^2)^4\, 10^8 =$$

$$= (-1)^4\, 10^8 = 10^8$$

$$(i\,10)^8 + 10^8 = 2 \cdot 10^8 > 0$$

Per h suff. grande, γ_h non racchiude nessuno degli zeri di

$$p(z) = z^8 + 10^8$$

$\Rightarrow \exists\, H > 10$ t.c. $\forall\, h \geq H$

$$\int_{\gamma_h} \frac{dz}{z^8 + 10^8} = 0$$

$$\Rightarrow \lim_{h \to +\infty} \int_{\gamma_h} " = 0$$

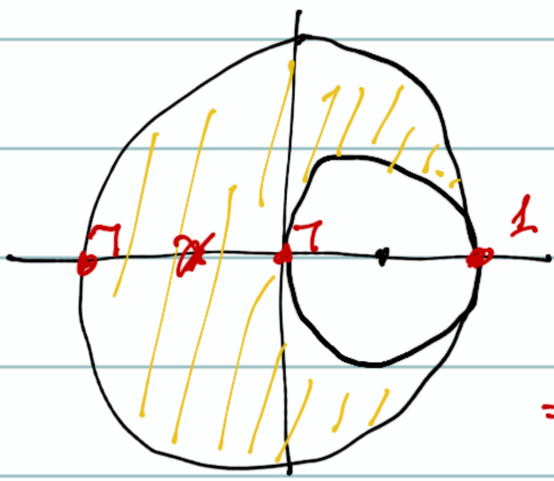
$\square$

$f(0) = 0$

$$f(z) = \frac{z}{z-1} \qquad f(-1) = \frac{-1}{-1-1} = \frac{1}{2}$$

Determinare l'immagine mediante f dell'ins.

$E = B_1(0) \setminus \overline{B_{1/2}\left(\frac{1}{2}\right)}$



$f(E) = \{ w \in \mathbb{C} \mid w = f(z),\ z \in E \} =$

$= \{ w \in \mathbb{C} \mid f^{-1}(w) \in E \}$

$$f(E) = \left\{ w \in \mathbb{C} \;\middle|\; |f^{-1}(w)| < 1,\ \left| f^{-1}(w) - \frac{1}{2} \right| > \frac{1}{2} \right\}$$

$$f(z) = \frac{z}{z-1}, \qquad f^{-1}(w) = \frac{-w}{-w+1} =$$

$$f^{-1}(w) = \frac{w}{w-1}$$

$$\left| \frac{w}{w-1} \right| < 1 \quad , \qquad |w| < |w-1|$$

$$|w|^2 < |w-1|^2$$

$$\cancel{|w|^2} < \cancel{|w|^2} \underbrace{- 2\,\mathrm{Re}\,w}_{} + 1$$

$$\boxed{\mathrm{Re}\,w < \frac{1}{2}}$$

$$\left| \frac{w}{w-1} - \frac{1}{2} \right| > \frac{1}{2}$$

$$\left| \frac{2w - w + 1}{\cancel{2}\underbrace{(w-1)}_{}} \right| > \frac{1}{\cancel{2}}$$

$$|w+1|^2 > |w-1|^2$$

$$\cancel{|w|^2} + 2\,\mathrm{Re}\,w + \cancel{1} > \cancel{|w|^2} \underbrace{- 2\,\mathrm{Re}\,w}_{} + \cancel{1}$$

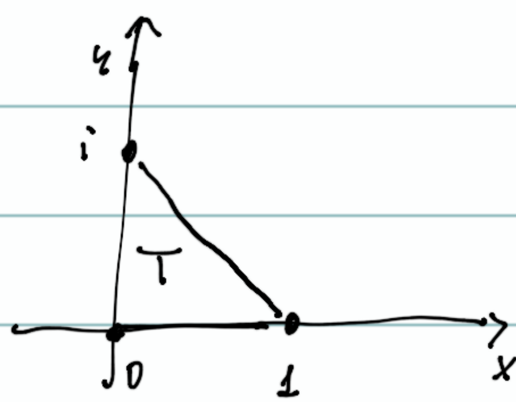
$\Leftarrow \operatorname{Re} w > 0$

$$f(E) = \{ w \in \mathbb{C} \mid 0 < \operatorname{Re} w < \tfrac{1}{2} \}$$

$$f(z) = \frac{z-1}{z+1}$$

T il triangolo di vertici $0, 1, i$.

Det: $f(T)$



$f(\mathbb{R}) = \mathbb{R}$, $f(i\mathbb{R})$?

$$f(iy) = \frac{iy-1}{iy+1} = \frac{(iy-1)(-iy+1)}{1+y^2} =$$

$$= \frac{y^2+iy+iy-1}{1+y^2} = \frac{y^2-1+2iy}{1+y^2}$$

$$|f(iy)|^2 = 1$$

$$(y^2-1)^2 = y^4-2y^2+1$$

$$(2y)^2 = 4y^2$$

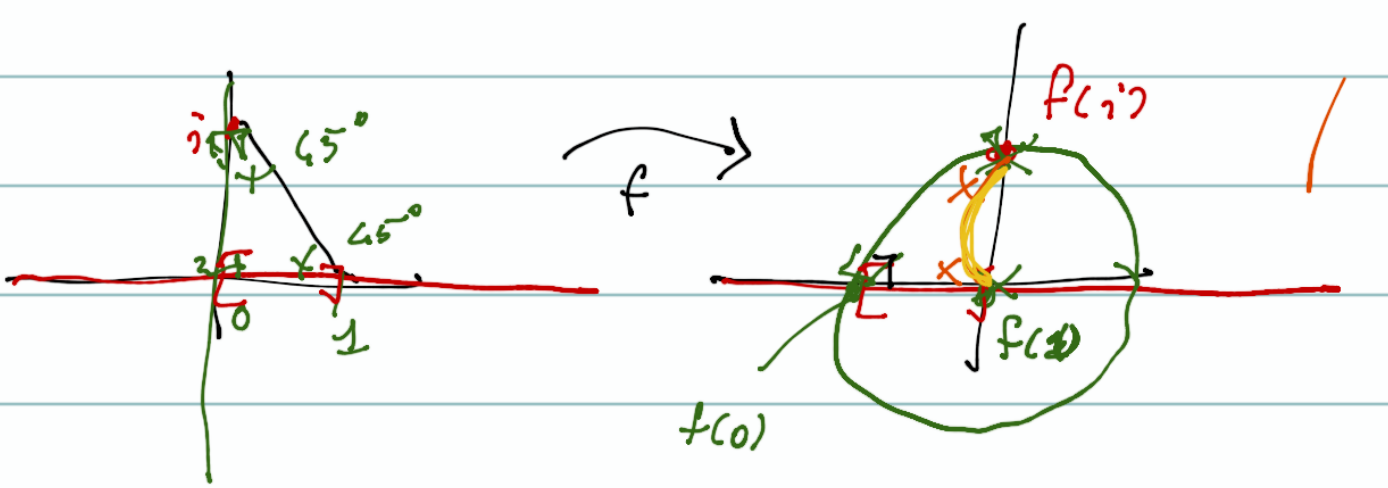
$$(y^2-1)^2+(2y)^2 = (y^2+1)^2$$

$$f(i\mathbb{R}) = \partial B_1(0)$$

$$f(z) = \frac{z-1}{z+1}$$

$$f(0) = -1$$

$$f(i) = \frac{i-1}{i+1} = i$$



$$\ell = \{x+y=1\} \longrightarrow f(\ell) = ?$$

$$f(z) = \frac{z-1}{z+1}$$

$$f^{-1}(w) = \frac{w+1}{-w+1} \quad , \quad f(\ell) = \{w \mid f^{-1}(w) \in \ell\}$$

$$\mathrm{Re}\left(f^{-1}(w)\right) + \mathrm{Im}\left(f^{-1}(w)\right) = 1$$

$$\mathrm{Re}\left(\frac{w+1}{-w+1}\right) + \mathrm{Im}\left(\frac{w+1}{-w+1}\right) = 1$$

$$\frac{1}{2}\left(\frac{w+1}{-w+1} + \frac{\bar{w}+1}{-\bar{w}+1}\right) + \frac{1}{2i}\left(\frac{w+1}{-w+1} - \frac{\bar{w}+1}{-\bar{w}+1}\right) = 1$$

$$\frac{1}{2}\left(\frac{(w+1)(-\bar{w}+1) + (-w+1)(\bar{w}+1)}{1+|w|^2}\right) +$$

$$+ \frac{1}{2i}\,\frac{(w+1)(-\bar{w}+1) - (-w+1)(\bar{w}+1)}{1+|w|^2} = 1$$

$$\frac{1}{2}\left(\underbrace{-|w|^2 + \cancel{w} - \cancel{\overline{w}} + 1 - |w|^2 - \cancel{w} + \cancel{\overline{w}} + 1}\right) +$$

$$+ \frac{1}{2i}\left(-\cancel{|w|^2} + w - \overline{w} + \cancel{1} + \cancel{|w|^2} + w - \overline{w} - \cancel{1}\right) = 1 + |w|^2$$

$$\frac{1}{2}\left(2 - 2|w|^2\right) + \frac{1}{2i}\left(2w - 2\overline{w}\right) = 1 + |w|^2$$

$$\cancel{1} - |w|^2 + 2\,\mathrm{Im}\,w = \cancel{1} + |w|^2$$

$$\boxed{2|w|^2 - 2\,\mathrm{Im}\,w = 0}$$