$$\lim_{R \to +\infty} \int_{-R}^{R} \frac{x^3 e^{-ix}}{x^4+1}\, dx$$

$$f(z) = \frac{z^3}{z^4+1} \qquad |f(z)| \to 0 \qquad |z| \to \infty$$

$$\int_{-R}^{R} \frac{x^3 e^{-ix}}{x^4+1}\, dx \underset{t=-x}{=} \int_{R}^{-R} \frac{-t^3 e^{it}}{t^4+1} (-dt) =$$

$$= -\int_{-R}^{R} \frac{t^3 e^{it}}{t^4+1}\, dt$$

$$g(z) = -\frac{z^3}{z^4+1}$$

$$\lim_{R \to +\infty} \int_{-R}^{R} g(t)\, e^{it}\, dt =$$

$$= 2\pi i \sum_{\operatorname{Im} z_k > 0} \operatorname{Res}\left(g(z) e^{iz}, z_k\right)$$

g ha poli dove $z^4+1 = 0$

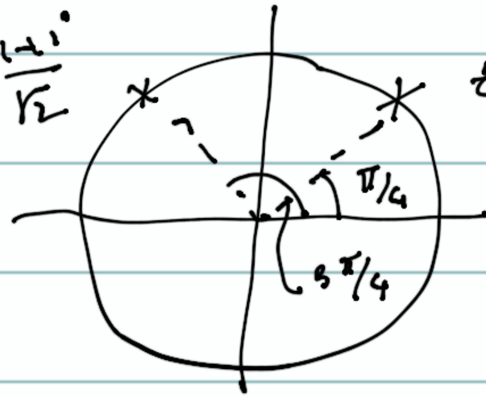
$$z^2 = \pm i = \pm e^{i\pi/2}$$

$$z_k = e^{i\frac{\pi}{4} + i\frac{2k\pi}{4}} \qquad k = 0, 1, 2, 3$$

z_0, z_1 stanno nel semipiano sup.

$$z_0 = e^{i\pi/4}, \quad z_1 = e^{i3\pi/4}$$

$z_1 = \frac{-1+i}{\sqrt{2}}$ $\qquad$ $z_0 = \frac{1+i}{\sqrt{2}}$



$$\mathrm{Res}(g(z)e^{iz}, z_0) = -\frac{z_0^3}{4z_0^3} e^{iz_0} = -\frac{1}{4} e^{i\left(\frac{1+i}{\sqrt{2}}\right)} =$$

$$= -\frac{1}{4} e^{-\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}}$$

$$\mathrm{Res}(g(z)e^{iz}, z_1) = -\frac{z_1^3}{4z_1^3} e^{iz_1} = -\frac{1}{4} e^{i\left(\frac{-1+i}{\sqrt{2}}\right)} =$$

$$= -\frac{1}{4} e^{-\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}}$$

$$\lim_{R\to+\infty} \int_{-R}^{R} \text{''} = \frac{2\pi i}{4\,2}\left(-e^{-\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}} - e^{-\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}}\right)$$

$$= - \frac{\pi i}{2} e^{-\frac{1}{\sqrt{2}}}\left(e^{i\frac{1}{\sqrt{2}}} + e^{-i\frac{1}{\sqrt{2}}}\right) =$$

$$= \pi\, e^{-\frac{1}{\sqrt{2}}}\ \frac{e^{i\frac{1}{\sqrt{2}}}+e^{-i\frac{1}{\sqrt{2}}}}{2i} =$$

$$= \frac{1}{i}\,\pi\, e^{-\frac{1}{\sqrt{2}}}\cos\frac{1}{\sqrt{2}}$$

$$f(z) = \frac{e^{2z}}{81} + \underline{\underline{(z+i)^5}}$$

quanti zeri ha in $B_2(0)$?

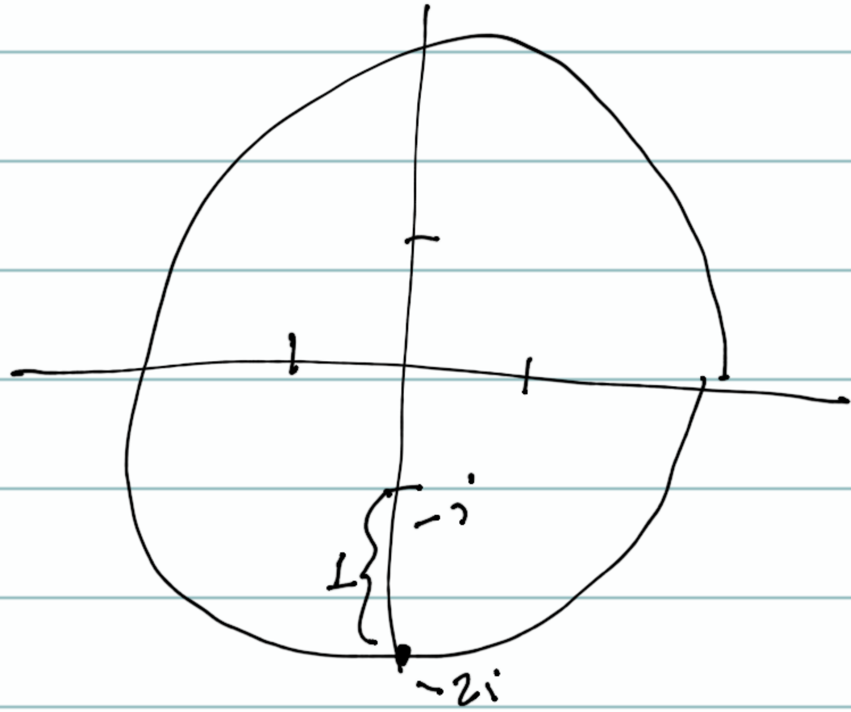
Poniamo $g(z) = (z+i)^5$

g ha uno zero di molt. 5 in $-i \in B_2(0)$

$$(f-g)(z) = \frac{e^{2z}}{81}$$

$$|z|=2,\quad |f-g| \le \frac{e^{2\cdot 2}}{81} = \left(\frac{e}{3}\right)^4 < 1$$

$$|g(z)|_{|\partial B_2(0)} = |z+i|^5_{|\partial B_2} \geq 1$$



Abbiamo

$$|f-g| < |g| \quad \text{su} \quad \partial B_2(0)$$

Quindi f ha in $B_2(0)$, 5 zeri contati con la molteplicità.

— · —

Caratterizzare le trasf. di Möbius che trasformano in sé $\partial B_2(0)$.

Sia $\varphi(z) = \dfrac{az+b}{cz+d}$ t.c. $\varphi(\partial B_2(0)) = \partial B_2(0)$

Due casi:

i) $\varphi(B_2(0)) = B_2(0)$

ii) $\varphi(B_2(0)) = \mathbb{C} \setminus (\overline{B_2(0)}) \cup \{\infty\}$.

<u>Caso i)</u> φ è autom di $B_2(0)$ in sé.

Sia $\lambda : B_1(0) \to B_2(0)$ data da

$$\lambda(z) = 2z$$

$$B_1 \xrightarrow{\lambda} B_2 \xrightarrow{\varphi} B_2 \xrightarrow{\lambda^{-1}} B_1$$

$$\underline{\lambda^{-1} \circ \varphi \circ \lambda(z) = e^{i\theta} \frac{z-c}{1-\bar{c}z}} \qquad \theta \in \mathbb{R}, \quad c \in B_1(0)$$

$$\varphi \circ \lambda(z) = 2e^{i\theta} \frac{z-c}{1-\bar{c}z}$$

$$\varphi(w) = 2e^{i\theta} \frac{\frac{w}{2}-c}{1-\bar{c}\,w/2}$$

$$\boxed{\begin{array}{l} \underline{\lambda(z) = w} \\ 2z = w \\ z = \frac{w}{2} \end{array}}$$

$$\boxed{\varphi(w) = 2e^{i\vartheta}\,\frac{w-2c}{2-\bar{c}w}}$$

Tipo (i)?

Caso ii $\quad \eta(z) = \dfrac{4}{z}$

$$\eta : B_2(0) \to \mathbb{C} \setminus \left(\overline{B_2(0)}\right) \cup \{\infty\}$$

Se $\quad \varphi : B_2(0) \to \mathbb{C} \setminus \left(\overline{B_2(0)}\right) \cup \{\infty\}$

$\psi = \eta \circ \varphi : B_2(0) \to B_2(0) \quad$ è di tipo 1.

$$\varphi = \eta \circ \psi$$

$$\varphi(z) = \eta\left(2e^{i\vartheta}\,\frac{z-2c}{2-\bar{c}z}\right) =$$

$\vartheta \in \mathbb{R}$

$|c| < 1$

$$= 4\,\frac{1}{2}\,e^{-i\vartheta}\,\frac{2-\bar{c}z}{z-2c} =$$

$$= 2\,e^{-i\vartheta}\,\frac{2-\bar{c}z}{z-2c} \qquad \text{Tipo (iii).}$$

$$\frac{1}{2\pi i}\int_{\partial B_1(0)} (z^5+z^4+z^3)\,e^{-\frac{1}{z}}\,dz$$

$$e^{-\frac{1}{z}} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!}\,\frac{1}{z^k}$$

$$z^5 e^{-\frac{1}{z}} = \dots + \frac{(-1)^6}{6!}\,\frac{1}{z} + \dots$$

$$z^4 e^{-\frac{1}{z}} = \dots + \frac{-1}{5!}\,\frac{1}{z} + \dots$$

$$z^3 e^{-\frac{1}{z}} = \dots + \frac{1}{4!}\,\frac{1}{z} + \dots$$

$$\operatorname{Res}\left((z^5+z^4+z^3)\,e^{-\frac{1}{z}}, 0\right) =$$

$$= \frac{1}{6!} - \frac{1}{5!} + \frac{1}{4!} = \frac{1}{4!}\left(\frac{1}{30} - \frac{1}{5} + 1\right) =$$

$$= \frac{1}{24}\,\frac{1-6+30}{30} = \frac{\cancel{25}^{\,5}}{24 \cdot \cancel{30}_{\,6}}$$

$$\frac{1}{2\pi i}\int_{\partial B_1(0)} (\dots)\, e^{-\frac{1}{z}}\, dz = \mathrm{Res}(\dots, 0) = \frac{5}{24 \cdot 6}$$

$$\boxed{v(z) = e^{3x}\sin 3y - \cos 5x \sinh 5y}$$

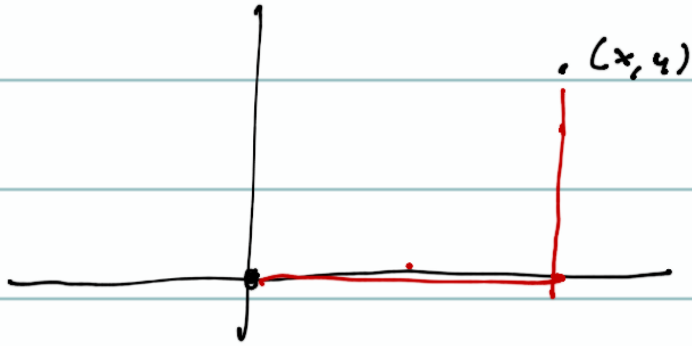
$$u(z) = e^{3x}\cos 3y + \cosh 5y \sin 5x$$

$z = x + iy \in \mathbb{C}$.

Stabilire se u è armonica, e se sì, determinarne un'arm. coniugata.

$$\begin{cases} v_x = -u_y \\ v_y = u_x \end{cases}$$

$$dv = -u_y\, dx + u_x\, dy$$



$$e^{5x}\cos 3y = \mathrm{Re}\left(e^{3(x+iy)}\right) =$$

$$= \mathrm{Re}\; e^{3z}$$

la coniug di $e^{3x}\cos 3y$ è $e^{3x}\sin 3y$

$$\boxed{\cosh 5y \,\sin 5x}$$

$$\sinh(\alpha+\beta) = \cosh\alpha \sinh\beta + \sinh\alpha \cosh\beta$$

$$\sinh(x+iy) = \cosh x \sinh(iy) + \sinh x \cos y =$$

$$= -\underbrace{\cosh x \, i \sin y} + \sinh x \cos y$$

$$\cosh x \sin y = - \,\mathrm{Im}\left(\sinh(x+iy)\right).$$

$$\underbrace{\cosh 5y \sin 5x} = -\,\mathrm{Im}\left(\sinh(5y + i5x)\right) =$$

$$= -\,\mathrm{Im}\left(\sinh i(5x - i5y)\right) =$$

$$= -\,\mathrm{Im}\left(i \sin(5x - i5y)\right) =$$

$$= \mathrm{Re}\left(\sin(5x - i5y)\right) =$$

$$= \mathrm{Re}\left(\sin 5\bar{z}\right) = \mathrm{Re}\left(\overline{\sin 5z}\right) =$$

$$= \mathrm{Re}\left(\sin 5z\right)$$

il coniugato di $\cosh 5y \sin 5x$ è :

$$\mathrm{Im}(\sin 5z) =$$

$$= \mathrm{Im}\left(\sin 5(x+iy)\right) =$$

$$= \mathrm{Im}\left(\underline{\sin 5x \cos(5iy)} + \underline{\cos 5x \sin(5iy)}\right)$$

$$= -\cos 5x \sinh 5y$$

Sia γ la curva:

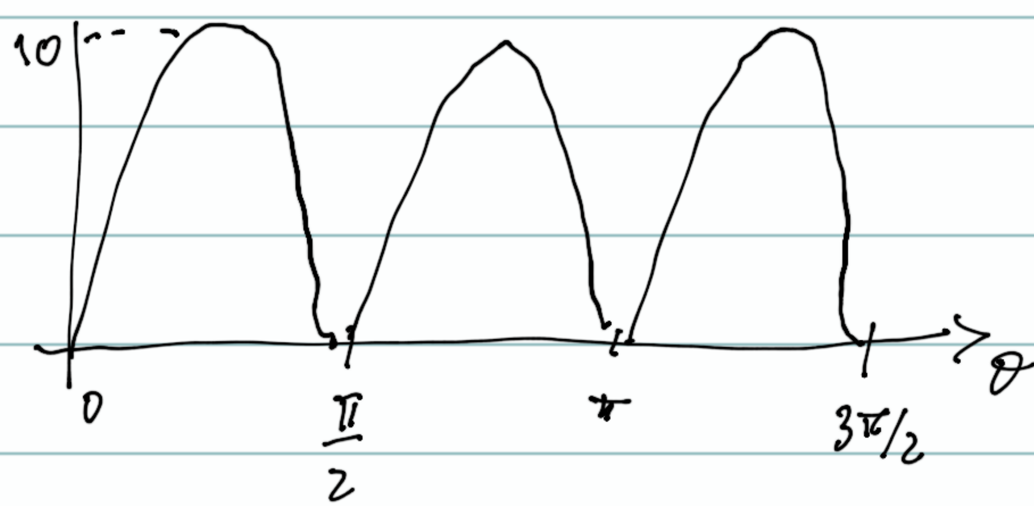
$$z = 10\,|\sin 2\vartheta|\,e^{i\vartheta} \quad , \quad 0 \le \vartheta \le \frac{3\pi}{2}$$

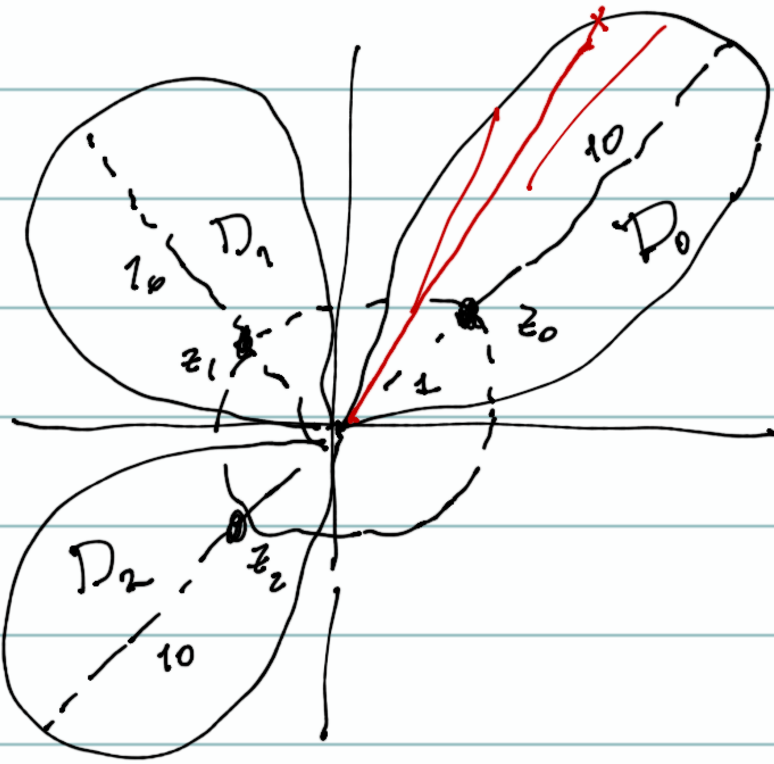
Calcolare

$$\int_\gamma \frac{1}{z^4+1}\,dz$$

— . —

$$z = r(\vartheta)\,e^{i\vartheta} \quad , \quad r(\vartheta) = 10\,|\sin 2\vartheta|$$





Quindi $\quad \gamma = \partial D_0 + \partial D_1 + \partial D_2$

dove $\quad D_0 = \left\{ z \in \mathbb{C} \;\middle|\; z = re^{i\vartheta},\ 0 \leq \vartheta \leq \frac{\pi}{2},\ 0 < r < 10\,|\sin 2\vartheta| \right\}$

$D_1 = \{ \;\text{"}\; | \;\text{"}\; \frac{\pi}{2} \leq \vartheta \leq \pi,\ 0 < r < \;\text{"}\; \}$

$D_2 = \{ \;\text{"}\; | \;\text{"}\; \pi \leq \vartheta \leq \frac{3}{2}\pi,\ 0 < r < \;\text{"}\; \}.$

$$\int_\gamma 〃 = \sum_{k=0}^{2} \mathrm{Res}\left(\frac{1}{z^4+1}, z_k\right) 2\pi i$$

$$= 2\pi i \left(\frac{1}{3z_0^3} + \frac{1}{3z_1^3} + \frac{1}{3z_2^3} \right)$$

$z_0 = e^{i\pi/4}$ $\quad$ $z_0^3 = e^{i\frac{3}{4}\pi}$

$z_1^3 = \cdots$, $z_2^3 = \cdots$