MASTER DEGREE COURSE IN MATHEMATICS, A.Y. 2018/19 ADVANCED GEOMETRY 3 - PROBLEMS FOR THE ORAL EXAM

1. Let $X \subset \mathbb{P}^4$ be a hypersurface of degree $d \geq 2$. Prove that, if X contains a plane, then it has singular points.

2. Let $Z \subset \mathbb{G}(1,3) \times \mathbb{G}(1,3)$ be the set defined as follows:

$$Z = \{(\ell_1, \ell_2) \mid \ell_1 \cap \ell_2 \neq \emptyset\}.$$

Prove that Z is closed and find its equations.

3. Let $S = K[x_0, \ldots, x_3]$ be the polynomial ring in 4 variables, let S_4 be its homogeneous component of degree 4. Prove that the set $X \subset \mathbb{P}(S_4)$ of proportionality classes of reducible polynomials is a proper closed subset, find its irreducible components and their dimensions.

4. Prove that $\mathbb{A}^1_{\mathbb{C}} \setminus \{point\}$ and $\mathbb{P}^1_{\mathbb{C}} \setminus \{point\}$ are not isomorphic to any projective variety.

5. Fix coordinates $[t_0, t_1]$ in $\mathbb{P}^1$ and $[x_0, x_1, x_2]$ in $\mathbb{P}^2$. In $\mathbb{P}^1 \times \mathbb{P}^2$ consider the closed subset $Z = V(t_0 x_0^2 - t_1 x_1^2)$, with the two projections p_1, p_2 to $\mathbb{P}^1$ and $\mathbb{P}^2$ respectively. Give a description of the fibres of p_1 and find the singular fibres, if any. Let $E_2[0,0,1] \in \mathbb{P}^2$. Prove that the points of $\mathbb{P}^1 \times \{E_2\}$ are singular points of Z.

6. Let $f : X \to Y$ be a finite surjective morphism of projective varieties. Let $Z \subset X$, $Z \neq X$ be a closed subset. Prove that $f(Z) \neq Y$.

7. Sia $X \subset \mathbb{A}^3_{\mathbb{C}}$ la curva le cui componenti irriducibili sono i tre assi coordinati. Scrivere equazioni per X e provare che l'ideale di X non può essere generato da due soli polinomi.

8. Siano L, M, N le seguenti rette nello spazio proiettivo $\mathbb{P}^3$:

$$L : x_0 = x_1 = 0, \; M : x_2 = x_3 = 0, \; N : x_0 - x_2 = x_1 - x_3 = 0.$$

Descrivere la sottovarietà di $\mathbb{G}(1,3)$ parametrizzante le rette che intersecano L, M, N. Sia $X \subset \mathbb{P}^3$ l'unione di tali rette: scrivere equazioni per X.