(1)

$$\int_{(0,+\infty)} \underbrace{\frac{e^{-x^2}}{1+(x-n)^2}}_{f_n(x)} \, dx$$

$$0 < f_n(x) \leq e^{-x^2} \quad \text{integrabile.}$$

$$\lim_{n\to\infty} f_n(x) = 0 \qquad \forall x$$

Per il teor di conv. dominata

$$\lim \int_{(0,+\infty)} f_n(x)\, dx = 0$$

(2) (a)

$$\nu(E) = \int_E f \, d\mu \overset{\text{Hölder}}{\leq}$$

$$\leq \|\chi_E\|_{p'} \|f\|_p = \mu(E)^{1/p'} \|f\|_p$$

(b) $\|f\|_p^p = \int |f|^p \, d\mu =$

$$= \int_0^{+\infty} \mu\{|f|^p > t\} \, dt =$$

$$= \int_0^{+} \mu\{|f| > t^{1/p}\} \, dt \qquad \forall p.$$

Sia $E_s = \{|f| > s\}$, $\mu(s) = \mu(E_s)$

$$\|f\|_q^q = \int_0^{+\infty} \mu(t^{1/q}) \, dt \underset{s = t^{1/q}}{=}$$

$$= \int_0^{+\infty} \mu(s) \, q \, s^{q-1} \, ds$$

So $\nu(E_s) = \int_{E_s} f \le \mu(E_s)^{1/p'}$, Per Chebishev

$$\mu(E_s) \le \frac{1}{s} \nu(E_s) \le \frac{1}{s} \mu(E_s)^{1/p'}$$

$$\mu(E_s)^{1/p} \le \frac{1}{s}$$

$$\|f\|_q^q \leq \int_0^1 \mu(X)\, q\, s^{q-1}\, ds +$$

$$+ \int_1^{+\infty} q\, s^{-p+q-1}\, ds < \infty. \quad \square$$

(3) Sia $X = \mathbb{R}^n \setminus E$, $f : X \to \mathbb{R}$ è continua quindi $\forall\, A \subset \mathbb{R}$ aperto

$$f^{-1}(A) \cap X = B \cap X, \quad B \text{ aperto}$$

in $\mathbb{R}^n$

$$f^{-1}(A) = \underbrace{(B \cap X)}_{\mathcal{L}} \cup \underbrace{(E \cap f^{-1}(A))}_{\subset E,\ \mu(E)=0}$$

quindi $f^{-1}(A) \in \mathcal{L}$.

— Più in dettaglio: sia $x \in f^{-1}(A) \cap X$ e sia $y = f(x) \in A$

$\forall \varepsilon > 0\ \exists \delta > 0$ t.c. se $|x' - x| < \delta$, allora $|f(x') - f(x)| < \varepsilon$. Sia $\varepsilon > 0$

t.c. $B_\varepsilon(y) \subset A$ allora

$$f(B_\delta(x)) \subset B_\varepsilon(y) \subset A$$

cioè

$$B_\delta(x) \subset f^{-1}(A)$$

$$B_\delta(x) \cap X \subset f^{-1}(A) \cap X$$

Quindi $\delta = \delta(x)$, allora

$$B = \bigcup_{x \in f^{-1}(A) \cap X} B_{\delta(x)}(x) \quad \text{è}$$

aperto in $\mathbb{R}^n$ e $f^{-1}(A) \cap X \subseteq$

$$\subseteq B \cap X \subseteq f^{-1}(A) \cap X .$$

$\square$