① $f(z) = \dfrac{10-z}{1-10z} = \dfrac{z-10}{10z-1}$

$f^{-1}(w) = \dfrac{-w+10}{-10w+1} = \dfrac{w-10}{10w-1} = f(w)$

④ $f(B_1(0)) = \{ w \mid |f^{-1}(w)| < 1 \}$

$\left| \dfrac{w-10}{10w-1} \right| < 1 \quad , \quad |w-1|^2 < |10w-1|^2$

$100|w|^2 - 200 \operatorname{Re} w + \cancel{1} > |w|^2 - 2 \operatorname{Re} w + \cancel{1}$

$99|w|^2 - 198 \operatorname{Re} w > 0$

$|w|^2 - 2 \operatorname{Re} w > 0$

$|w-1|^2 > 1$

$$f(B_1(0)) = \mathbb{C} \setminus \left(\overline{B_1(1)} \right).$$

(b) $$\ell = \{ \mathrm{Re}\, w = 1 \}$$

$$f^{-1}(\ell) = \{ z \mid \mathrm{Re}\, f(z) = 1 \}$$

$$\mathrm{Re}\left(\frac{10 - z}{1 - 10z} \right) = 1$$

$$\frac{1}{2}\left(\frac{10 - z}{1 - 10z} + \frac{10 - \bar{z}}{1 - 10\bar{z}} \right) = 1$$

$$\frac{1}{2} \frac{(10 - z)(1 - 10\bar{z}) + (10 - \bar{z})(1 - 10z)}{|1 - 10z|^2} = 1$$

$$10 - z + 10|z|^2 - 100\bar{z} + 10 - \bar{z} - 100z + 10|z|^2 =$$

$$= 2$$

$$20|z|^2 - 101\, z - 101\, \bar{z} + 20 = 2$$

$$10|z|^2 - 2\cdot 101\ \mathrm{Re}\, z + 18 = 0$$

$$|z|^2 - 2\,\frac{101}{10}\,\mathrm{Re}\, z + \Big(\underset{\substack{\uparrow\\ c}}{\frac{101}{10}}\Big)^2 = \underbrace{\left(\frac{101}{10}\right)^2 - 18}_{R^2}$$

$$z \in \partial B_R(-c),$$

(2) $$u(z) = r^{4/3}\cos\frac{4}{3}\vartheta$$

$$f(z) = r^{4/3}\left(\cos\frac{4}{3}\vartheta + i\sin\frac{4}{3}\vartheta\right) =$$

$$= \exp\left(\frac{4}{3}\,\mathrm{Log}\, z\right) \in H(S)$$

Infatti in S : $\mathrm{Log}\, z = \log r + i\vartheta$

quindi $\frac{4}{3}\mathrm{Log}\, z = \frac{4}{3}\log r + i\,\frac{4}{3}\vartheta.$

$$v(z) = r^{4/3} \sin \frac{4}{3}\vartheta .$$

(3) $$f(z) = \frac{z^4}{(z^2+1)(z^4+1)}$$

$$|f(z)| \le \frac{1}{|z^2+1|} \le \frac{1}{|z|^2-1} = O(|z|^{-2})$$

$|z| \to \infty$,

se $|z| > 1$

Quindi

$$\int_{-\infty}^{\infty} f(t)\,dt = 2\pi i \sum_{\operatorname{Im} z_k > 0} \operatorname{Res}(f, z_k)$$

f ha poli semplici dove

$$z^2+1=0 \qquad z = \underbrace{i}_{z_1}, -i$$

e dove

$$z^4+1=0$$

$$z = \underbrace{\frac{1+i}{\sqrt{2}}}_{z_2}, \underbrace{\frac{-1+i}{\sqrt{2}}}_{z_3}, \frac{-1-i}{\sqrt{2}}, \frac{1-i}{\sqrt{2}}$$

$$\operatorname{Res}(f, i) = \frac{z_1^4}{2z_1(1+z_1^4) + \underbrace{(1+z_1^2)}_{\substack{\shortparallel \\ 0}} 4z_1^3} =$$

$$= \frac{1}{2i(1+1)} = \frac{1}{4i}$$

$$\operatorname{Res}(f, z_i) = \frac{z_i^4}{(1+z_i^2)\, 4 z_i^3} \qquad i = 2,3$$

$$= \frac{z_i}{4(1+z_i^2)}$$

$$z_2^2 = \left(\frac{1+i}{\sqrt{2}}\right)^2 = i$$

$$z_3^2 = \left(\frac{-1+i}{\sqrt{2}}\right)^2 = \left(\frac{1-i}{\sqrt{2}}\right)^2 = -i$$

$$\operatorname{Res}(f, z_2) = \frac{\frac{1+i}{\sqrt{2}}}{4(1+i)} = \frac{1}{4\sqrt{2}}$$

$$\operatorname{Res}(f, z_3) = \frac{\frac{-1+i}{\sqrt{2}}}{4(1-i)} = -\frac{1}{4\sqrt{2}}$$

$$\int_{-\infty}^{+\infty} f(t)\, dt = 2\pi i \operatorname{Res}(f, i) = \frac{2\pi i}{4i} = \frac{\pi}{2}.$$